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ON THE RELATION OF LATTICE REPLETENESS AND C-REAL COMPACTNESS

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Consider any set X and any two lattices of subsets of X, $\mathscr{L}_1$, $\mathscr{L}_2$, such that $\mathscr{L}_1 \subset \mathscr{L}_2$. It is well-known that, in case $\mathscr{L}_1$ semiseparates $\mathscr{L}_2$, there is a natural mapping from the Wallman space $IR(\mathscr{L}_2$ to the Wallman space $IR(\mathscr{L}_1)$, namely, the restriction mapping. This paper deals with three situations, each involving two lattices of subsets of X, $\mathscr{L}_1$, $\mathscr{L}_2$, such that either $\mathscr{L}_1 \subset \mathscr{L}_2$, but $\mathscr{L}_1$ does not semiseparate $\mathscr{L}_2$, or $\mathscr{L}_1 \not\subset \mathscr{L}_2$, and for which there exists a continuous mapping ϕ from $IR(\mathscr{L}_2)$ to $IR(\mathscr{L}_1)$. Several properties of this function are discovered. By considering the restriction of ϕ to $IR(\sigma, \mathscr{L}_2)$, the set of countably additive elements of $IR(\mathscr{L}_2)$, several repleteness interrelations between the two lattices are obtained. The work culminates with the determination of the relationship between repleteness and c-realcompactness. Numerous topological measure theory—applications are given throughout the paper.

INTRODUCTION

Consider the following : any set X; any lattice of subsets of X, $\mathscr{L}$, such that $\phi, X \in \mathscr{L}$; the set whose general element is a $(0-1)$-valued, $\mathscr{L}$-regular, finitely additive measure on the algebra of subsets of X generated by $\mathscr{L}$, $IR(\mathscr{L})$, and also the Wallman topology on $IR(\mathscr{L})$. Now, consider any two such lattices $\mathscr{L}_1$, $\mathscr{L}_2$, with $\mathscr{L}_1 \subset \mathscr{L}_2$. It is well-known that, in case $\mathscr{L}_1$ semiseparates $\mathscr{L}_2$, there is a natural mapping from $IR(\mathscr{L}_2)$ to $IR(\mathscr{L}_1)$, namely, the restriction mapping. This situation has been investigated in great detail, especially with respect to various repleteness or completeness interrelations between the two lattices[2]. The more general situation in which either $\mathscr{L}_1 \subset \mathscr{L}_2$, but $\mathscr{L}_1$ does not semiseparate $\mathscr{L}_2$, or $\mathscr{L}_1 \not\subset \mathscr{L}_2$ has not been investigated in any generality.

In the first part of this paper, we consider three situations in which either $\mathscr{L}_1 \subset \mathscr{L}_2$, but $\mathscr{L}_1$ does not semiseparate $\mathscr{L}_2$, or $\mathscr{L}_1 \not\subset \mathscr{L}_2$, and for which there exists a continuous mapping ϕ from $IR(\mathscr{L}_2)$ to $IR(L_1)$. The basic idea is due to Zaicev[17],

who considered the special case of a Tychonoff space with $\mathscr{L}_1 = \mathscr{L}$ and $\mathscr{L}_2 = \mathscr{F}_R$, where $\mathscr{F}_R$ is the lattice of subsets of X generated by the collection of regular closed sets. (Zaicev uses the notation $IR(\mathscr{F}_R) = \omega_\pi X$; the notation $IR(\mathscr{L}) = \beta X$ is standard). We follow Zaicev's construction to a large extent for this special case, but we go beyond that, in both the general and the special case, by considering the restriction of the mapping ϕ to $IR(\sigma, \mathscr{L}_2)$, the set whose general element is an element of $IR(\mathscr{L}_2)$ which is countably additive. This part of our work leads to new repleteness interrelations. Our work culminates in the last part of the paper where we give a determination of the relationship between c-realcompactness and repleteness of $\mathscr{F}_R$.

We present along with the general theory numerous applications to specific topological lattices, thus obtaining some known results in a different manner and also obtaining new results.

The terminology and notation are fairly standard and are consistent with those of Wallman[16], Frolik[8], Nöbeling[14], Bachman and Stratigos[2], and others. In the first section, and for the convenience of the reader, we will repeat some of these and we will mention some basic facts which are used throughout the paper. We will follow this practice with respect to certain topological facts which are scattered in the literature, and for which it is difficult to give just a few references; we will present these facts in the sections to which they are most relevant.

§ 1. *Terminology and notation and some basic facts* (a) Consider any set X and any lattice of subsets of X, $\mathscr{L}$. We shall always assume, without loss of generality for our purposes, that $\phi, X \in \mathscr{L}$. The definitions of the following concepts are found in Bachman and Stratigos[2]. $\mathscr{L}$ is separating, disjunctive, regular, normal, Lindelöf. Now, consider any two such lattices $\mathscr{L}_1$, $\mathscr{L}_2$, with $\mathscr{L}_1 \subset \mathscr{L}_2$. The definitions of the following concepts are also found in Bachman and Stratigos[2] $\mathscr{L}_1$ semiseparates $\mathscr{L}_2$, $\mathscr{L}_1$ separates $\mathscr{L}_2$, $\mathscr{L}_2$ is $\mathscr{L}_1$-countably bounded, $\mathscr{L}_2$ is $\mathscr{L}_1$-countably paracompact.

Consider any topological space X. The collection of closed sets is denoted by $\mathscr{F}$, the collection of open sets bo $\mathscr{O}$, the collection of clopen sets by $\mathscr{C}$, and the collection of zero sets by $\mathscr{Z}$ A closed set F is said to be regular closed iff $F^{0-} = F$. The collection of regular closed sets is denoted by $\mathscr{R}(\mathscr{F})$ or by $\mathscr{R}(X)$. Note the union of any two regular closed sets is regular closed; however, the intersection of any two regular closed sets need not be regular closed. The lattice of subsets of X generated by $\mathscr{R}(X)$ (in the set-theoretic sense) is denoted by $\mathscr{L}(\mathscr{R}(X))$ or by $\mathscr{F}_R$. The Boolean algebra of subsets of X generated by $\mathscr{R}(X)$ is also denoted by $\mathscr{R}(X)$. An open set O is said to be regularopen iff $\bar{O}^0 = O$. Note the intersection of any two regular open sets is regular open; however, the union of any two regular open sets need not be regular open. The lattice of subsets of X generated by the collection of regular open sets is denoted by $\mathscr{O}_R$.

(b) For an arbitrary function f, the domain of f is denoted by D_f.

The set whose general element is the intersection of an arbitrary subset of $\mathscr{L}$ is denoted by $t\mathscr{L}$. The algebra of subsets of X generated by $\mathscr{L}$ is denoted by $\mathscr{A}(\mathscr{L})$.

(c) Consider any algebra of subsets of X, $\mathscr{A}$. A measure on $\mathscr{A}$ is defined to be a function μ, from $\mathscr{A}$ to R, such that μ is finitely additive and bounded (see Alexandroff[1], p. 567.) The set whose general element is a measure on $\mathscr{A}(\mathscr{L})$ is denoted by $M(\mathscr{L})$.

For an arbitrary element of $M(\mathscr{L})$, μ, the support of μ is defined to be $\cap \{L \in \mathscr{L} / |\mu|(L) = |\mu|(X)\}$ and is denoted by $S(\mu)$.

An element of $M(\mathscr{L})$, μ, is said to be $\mathscr{L}$-regular iff for every element of $\mathscr{A}(\mathscr{L})$, E, for every positive number, ϵ, there exists an element of $\mathscr{L}$, L, such that $L \subset E$ and $|\mu(E) - \mu(L)| < \epsilon$. The set whose general element is an element of $M(\mathscr{L})$ which is $\mathscr{L}$-regular is denoted by $MR(\mathscr{L})$. An element of $M(\mathscr{L})$, μ, is said to be $\mathscr{L}$-(σ-smooth) iff for every sequence in $\mathscr{A}(\mathscr{L})$, $<A_n>$, if $<A_n>$ is decreasing and $\lim_n A_n = \phi$, then $\lim_n \mu(A_n) = 0$. The set whose general element is an element of $M(\mathscr{L})$ which is $\mathscr{L}$-(σ-smooth) is denoted by $M(\sigma, \mathscr{L})$. The set whose general element is an element of $M(\mathscr{L})$ which is $\mathscr{L}$-(σ-smooth) just for $<A_n>$ in $\mathscr{L}$ is denoted by $M(\sigma^*, \mathscr{L})$. Note for every element of $MR(\mathscr{L})$, μ, $\mu \in M(\sigma, \mathscr{L})$ iff $\mu \in M(\sigma^*, \mathscr{L})$.

The set whose general element is an element of $M(\mathscr{L})$, μ, such that $\mu(\mathscr{A}(\mathscr{L})) = \{0, 1\}$, that is, the set whose general element is a $(0 - 1)$-valued, finitely additive measure on $\mathscr{A}(\mathscr{L})$, is denoted by $I(\mathscr{L})$. For an arbitrary element of $\mathscr{A}(\mathscr{L})$, A, $\{\mu \in IR(\mathscr{L}) / \mu(A) = 1\}$ is denoted by $W(A)$. $\{W(L) / L \in \mathscr{L}\}$ is denoted by $W(\mathscr{L})$.

(d) $\mathscr{L}$ is said to be replete iff for every element of $IR(\mathscr{L})$, μ, if $\mu \in IR(\sigma, \mathscr{L})$, then $S(\mu) \neq \phi$.

(e) We note that there exists a one-to-one correspondence between $I(\mathscr{L})$ and the set of all prime $\mathscr{L}$-filters and there exists a one-to-one correspondence between $IR(\mathscr{L})$ and the set of all $\mathscr{L}$-ultrafilters. (Details can be found in Cohen[5]). It follows, therefore, that for every element of $I(\mathscr{L})$, μ, there exists an element of $IR(\mathscr{L})$, ν, such that $\mu \leqslant \nu$ on $\mathscr{L}$. (The proof involves a filter-ultrafilter argument.)

Also, we note that for any two lattices of subsets of X, $\mathscr{L}_1$, $\mathscr{L}_2$, if $\mathscr{L}_1 \subset \mathscr{L}_2$, then for every element of $IR(\mathscr{L}_1)$, μ, there exists an element of $IR(\mathscr{L}_2)$, ν, such that $\nu |_{\mathscr{A}(\mathscr{L}_1)} = \mu$. (The proof involves a filter-ultrafilter argument).

§ 2. In this section, we lay the groundwork upon which our investigation of the relationship between repleteness and c-realcompactness is based.

Review the definition of the (complete) Boolean algebra generated by the collection of regular closed sets of a topological space.

Consider any topological space X. Further, consider the collection of regular closed sets and denote it by $\mathscr{R}(\mathscr{F})$ or by $\mathscr{R}(X)$. Note $\mathscr{R}(\mathscr{F})$ is partially ordered by $\subset$. Denote $\subset$ by $\leq$ and consider the system $(\mathscr{R}(\mathscr{F}), \leqq)$.

(i) Show $(\mathscr{R}(\mathscr{F}), \leqq)$ is a (complete) lattice. Consider any collection in $\mathscr{R}(\mathscr{F})$, $\{E_\lambda; \lambda \in \Lambda\}$.

(α) Consider $\overline{\cup \{E^0_\lambda; \lambda \in \Lambda\}}$. Note $\overline{\cup \{E^0_\lambda; \lambda \in \Lambda\}} \in \mathscr{R}(\mathscr{F})$ and

$\overline{\cup \{E^0_\lambda; \lambda \in \Lambda\}} = \sup \{E_\lambda; \lambda \in \Lambda\}$.

(β) Consider $\overline{(\cap \{E_\lambda; \lambda \in \Lambda\})^0}$. Note $\overline{(\cap \{E_\lambda; \lambda \in \Lambda\})^0} \in \mathscr{R}(\mathscr{F})$ and $\overline{(\cap \{E_\lambda; \lambda \in \Lambda\})^0} = \inf \{E_\lambda; \lambda \in \Lambda\}$.

(γ) Consequently $(\mathscr{R}(\mathscr{F}), \leqq)$ is a (complete) lattice. Denote this lattice by $\mathscr{L}$.

(ii) Show $\mathscr{L}$ is distributive. (see Walker[15], p. 45.)

(iii) Show $\mathscr{L}$ is complemented. Consider any element of $\mathscr{R}(\mathscr{L})$, E. Further, consider $\overline{E}'$. Note $\overline{E' \in} \mathscr{R}(\mathscr{F})$ and $E \vee \overline{E}' = X$ and $E \wedge \overline{E}' = \phi$. Hence $\overline{E}'$ is a complement of E. Hence $\mathscr{L}$ is complemented. Moreover, since $\mathscr{L}$ is distributive, for every element of $\mathscr{R}(\mathscr{F})$, E, $\overline{E}'$ is the only complement of E; for this reason, $\overline{E}'$ is called the complement of E; denote the complement of E by E^*.

Summarizing : $\mathscr{L}$ is a (complete) lattice and $\mathscr{L}$ is distributive and complemented. Hence $\mathscr{L}$ is a (complete) Boolean algebra. $\mathscr{L}$ is referred to as the (complete) Boolean algebra generated by $\mathscr{R}(\mathscr{F})$. This Boolean algebra is also denoted by $\mathscr{R}(\mathscr{F})$ or by $\mathscr{R}(X)$.

The following proposition refers to the Boolean algebra $\mathscr{R}(X)$.

Proposition 2.1—Consider $(IR(\mathscr{R}(X)), W(\mathscr{R}(X)))$. Then

(a) For every element of $\mathscr{R}(X)$. E, $W(E^*) = W(E)'$.

(b) For any two elements of $\mathscr{R}(X)$, E_1. E_2, $W(E_1 \vee E_2) = W(E_1) \cup W(E_2)$.

(c) For any two elements of $\mathscr{R}(X)$, E_1, E_2, $W(E_1 \wedge E_2) = W(E_1) \cap W(E_2)$.

PROOF : (a) Consider any element of $\mathscr{R}(X)$, E. Then, since $E \vee E^* = X$ and $E \wedge E^* = \phi$, for every element of $IR(\mathscr{R}(X))$, ν, $\nu(E) + \nu(E^*) = 1$. Hence for every element of $IR(\mathscr{R}(X))$, ν, $\nu \in W(E^*) \Leftrightarrow \nu(E^*) = 1 \Leftrightarrow \nu(E) = 0 \Leftrightarrow \nu \in W(E)'$. Hence $W(E^*) = W(E)'$.

(b) Consider any two elements of $\mathscr{R}(X)$, E_1, E_2. Further, consider any element of $IR(\mathscr{R}(X))$, ν.

Note

$$E_1 \vee E_2 = E_1 \vee (E_2 \wedge E_1^*) \text{ and } E_1 \wedge (E_2 \wedge E_1^*) = \phi.$$

Hence, since ν is additive, $\nu(E_1 \vee E_2) = \nu(E_1) + \nu(E_2 \wedge E_1^*)$.

Further, note $E_2 \wedge E_1^* = E_2 \wedge (E_1 \wedge E_2)^*$ and $E_1 \wedge E_2 \leqq E_2$.

Hence, since ν is subtractive, $\nu(E_2 \wedge E_1^*) = \nu(E_2) - \nu(E_1 \wedge E_2)$.

Consequently $\nu(E_1 \vee E_2) = \nu(E_1) + \nu(E_2) - \nu(E_1 \wedge E_2)$. Hence $\nu \in W(E_1 \vee E_2) \Leftrightarrow \nu(E_1 \vee E_2) = 1 \Leftrightarrow \nu(E_1) = 1$ or $\nu(E_2) = 1 \Leftrightarrow \nu \in W(E_1) \cup W(E_2)$. Hence $W(E_1 \vee E_2) = W(E_1) \cup W(E_2)$.

(c) (Proof omitted.)

The following four propositions are related to the description of the relationship between the collection of regular open sets of an arbitrary topological space and the collection of regular open sets of a subspace whose trace satisfies a certain "denseness" condition. (Although these propositions are generally known, we include them for the convenience of the reader, but omit their straightforward proofs.)

Consider the following : any topological space $(X, \mathscr{O})$; any subset of X, Y, such that Y is dense in X; the topological space $(Y, Y \cap \mathscr{O})$.

Proposition 2.2—For every element of $\mathscr{O}$, O, $\text{cl}_Y (Y \cap O) = Y \cap \text{cl}_X O$.

Proposition 2.3—For every subset of Y, A, $\text{int}_Y \text{cl}_Y A = Y \cap \text{int}_X \text{cl}_X A$.

Proposition 2.4— (i) For every regular open in X set O, $Y \cap O$ is regular open in Y.

(ii) For every regular open in Y set $\hat{O}$, there exists a regular open in X set O such that $\hat{O} = Y \cap O$.

Proposition 2.5—There exists a one-to-one correspondence between the collection of regular open in X sets and the collection of regular open in Y sets.

Observation : Since the complement of every regular open set is regular closed and vice versa, there exists a one-to-one correspondence between the collection of regular closed in X sets and the collection of regular closed in Y sets.

Next, we introduce a certain pair of Wallman spaces and a certain function relating these spaces and we devote the remainder of this section to the study of this function.

Consider the following : any set X; any lattice of subsets of X, $\mathscr{L}_1$, such that $\mathscr{L}_1$ is normal, separating, and disjunctive; any lattice of subsets of X, $\mathscr{L}_2$, such that

(1) $\iota\,\mathscr{L}_1 \subset \mathscr{L}_2$, or

(2) $\mathscr{L}_2 = \mathscr{L}\,(\mathscr{R}\,(\iota\,\mathscr{L}_1))$, or

(3) $\mathscr{L}_2 =$ the Boolean algebra $R\,(\iota\,\mathscr{L}_1)$;

the Wallman spaces $IR\,(\mathscr{L}_1)$, $IR\,(\mathscr{L}_2)$.

Now, consider any element of $IR\,(\mathscr{L}_2)$,ν. Further, consider $\{L_2 \in \mathscr{L}_2/\nu\,(L_2) = 1\}$ and set $\{L_2 \in \mathscr{L}_2/\nu\,(L_2) = 1\} = B_\lambda;\ \lambda \in \Lambda\}$.

Next, note for every λ, since $B_\lambda \subset X$ and $\mathscr{L}_1$ is separating and disjunctive $\bar{B}_\lambda{}^{IR(\mathscr{L}_1)}$ exists, (where $\bar{B}_\lambda{}^{IR(\mathscr{L}_1)}$ denotes the closure of B_λ in $IR\,(\mathscr{L}_1)$). Consider $\{\bar{B}_\lambda{}^{IR(\mathscr{L}_1)};\ \lambda \in \Lambda\}$.

Show $\cap\,\{\bar{B}_\lambda{}^{IR(\mathscr{L}_1)};\ \lambda \in \Lambda\} \neq \phi$. Since $IR\,(\mathscr{L}_1)$ is compact, it suffices to show $\{\bar{B}_\lambda{}^{IR(\mathscr{L}_1)};\ \lambda \in \Lambda\}$ has the Finite Intersection property (F. I. P.) Accordingly, consider any two elements of Λ, λ_1, λ_2, and show $\bar{B}_{\lambda_1}{}^{IR(\mathscr{L}_1)} \cap \bar{B}_{\lambda_2}{}^{IR(\mathscr{L}_1)} \neq \phi$. Note $B_{\lambda_1} \cap B_{\lambda_2} \subset \bar{B}_{\lambda_1}{}^{IR(\mathscr{L}_1)} \cap \bar{B}_{\lambda_2}{}^{IR(\mathscr{L}_2)}$ and $B_{\lambda_1} \cap B_{\lambda_2} \neq \phi$.

Hence

$$\bar{B}_{\lambda_1}{}^{IR(\mathscr{L}_1)} \cap \bar{B}_{\lambda_2}{}^{IR(\mathscr{L}_2)} \neq \phi.$$

Hence $\{\bar{B}_\lambda{}^{IR(\mathscr{L}_1)};\ \lambda \in \Lambda\}$ has the F.I. P. Consequently $\cap\,\{\bar{B}_\lambda{}^{IR(\mathscr{L}_1)};\ \lambda \in \Lambda\} \neq \phi$.

Show $\cap\,\{\bar{B}_\lambda{}^{IR(\mathscr{L}_1)};\ \lambda \in \Lambda\}$ is a singleton. Assume the countrary. Consider any two elements of $\cap\,\{\bar{B}_\lambda{}^{IR(\mathscr{L}_1)};\ \lambda \in \Lambda\}$, μ_1, μ_2 such that $\mu_1 \neq \mu_2$.

Observation : Consider any element of $\cap\,\{\bar{B}_\lambda{}^{IR(\mathscr{L}_1)};\ \lambda \in \Lambda\}$, μ. Now, consider any basic neighbourhood of μ,$W\,(A)'$. Then for every λ, since $\mu \in \cap\,\{\bar{B}_\lambda{}^{IR(\mathscr{L}_1)};\ \lambda \in \Lambda\}$, $B_\lambda \cap W\,(A)' \neq \phi$; hence, since $B_\lambda \subset X$, $(B_\lambda \cap X) \cap W\,(A)' \neq \phi$; hence $B_\lambda \cap (X \cap W\,(A)') \neq \phi$; hence, since $X \cap W\,(A)' = A'$, $B_\lambda \cap A' \neq \phi$; hence $B_\lambda \cap \overline{A'}^{\iota\mathscr{L}_1} \neq \phi$. Note $\overline{A'}^{\iota\mathscr{L}_1}$ is $\iota\,\mathscr{L}_1$-regular closed. Hence, since $\iota\,\mathscr{L}_1 \subset \mathscr{L}_2$, or $\mathscr{L}_2 = \mathscr{L}\,(\mathscr{R}\,(\iota\,\mathscr{L}_1))$, or $\mathscr{L}_2 =$ the Boolean algebra $\mathscr{R}\,(\iota\,\mathscr{L}_1)$, $\overline{A'}^{\iota\mathscr{L}_1} \in \mathscr{L}_2$.

Set $\overline{A'}^{t\mathscr{L}_1} = E$. Then $E \in \mathscr{L}_2$ and for every λ, $B_\lambda \cap E \neq \phi$. Since $\{B_\lambda ; \lambda \in \Lambda\} = \{L_2 \in \mathscr{L}_2 / \nu(L_2) = 1\}$ and $\nu \in IR(\mathscr{L}_2)$, $\{B_\lambda ; \lambda \in \Lambda\}$ is an $\mathscr{L}_2$-ultrafilter. Consequently $E \in \{B_\lambda ; \lambda \in \Lambda\}$. Consequently $\nu(E) = 1$.

Now, use the preceding observation as follows : Since $\mathscr{L}_1$ is normal and disjunctive, $IR(\mathscr{L}_1)$ is regular and T_2. Hence, since $\mu_1, \mu_2 \in IR(\mathscr{L}_1)$ and $\mu_1 \neq \mu_2$, there exist a basic neighbourhood of μ_1, $W(A_1)'$, and a basic neighbourhood of μ_2, $W(A_2)'$, such that $\overline{W(A_1)'} \cap \overline{W(A_2)'} = \phi$. Consider any such $W(A_1)'$, $W(A_2)'$. Then, by the preceding observation, for every λ

$$B_\lambda \cap \overline{A_i'}^{t\mathscr{L}_1} \neq \phi, \ (i = 1, 2).$$

Set

$$\overline{A_i'}^{t\mathscr{L}_1} = E_i, \ (i = 1, 2).$$

Then, by the observation, $\nu(E_i) = 1$, $(i = 1, 2)$. Further, note $\overline{W(A_i)'} \supset \overline{A_i'}^{IR(\mathscr{L}_1}$ $\supset \overline{A_i'}^{t\mathscr{L}_1} = E_i, (i = 1, 2)$. Hence, since $\overline{W(A_1)'} \cap \overline{W(A_2)'} = \phi$, $E_1 \cap E_2 = \phi$. Also, note, since $\nu(E_i) = 1$, $(i = 1, 2)$, $E_1 \cap E_2 \neq \phi$. Thus a contradiction has been reached. Consequently $\cap \{\bar{B}_\lambda^{IR(\mathscr{L}_1)} ; \lambda \in \Lambda\}$ is a singleton.

Now, set $\cap \{\bar{B}_\lambda^{IR(\mathscr{L}_1)} ; \lambda \in \Lambda\} = \{\mu\}$ and consider the function ϕ which is such that $D_\phi = IR(-_2)$ and for every element of $IR(\mathscr{L}_2)$, ν, $\phi(\nu) = \mu$.

Proposition 2.6—ϕ is continuous.

Proof : Consider any element of $IR(\mathscr{L}_2)$, ν_0, and show ϕ is continuous at ν_0. Set $\phi(\nu_0) = \mu_0$. Consider any basic neighbourhood of μ_0, $W(A)'$, and show there exists a neighbourhood of ν_0, V, such that $\phi(V) \subset W(A)'$. Since $IR(\mathscr{L}_1)$ is regular, there exists a basic neighbourhood of μ_0, $W(A_1)'$, such that $\overline{W(A_1)'} \subset W(A)'$. Consider any such $W(A_1)'$.

(i) Assume $t\mathscr{L}_1 \subset \mathscr{L}_2$. Then, since $A_1 \in L_1$, $A_1 \in \mathscr{L}_2$. Consider the element of $W(\mathscr{L}_2)$, $W(A_1)$. Then $W(A_1)'$ is $tW(\mathscr{L}_2)$-open.

(α) Show $W(A_1)'$ is a $tW(\mathscr{L}_2)$-neighbourhood of ν_0. Since $W(A_1)'$ is $t\,W(\mathscr{L}_2)$-open, it suffices to show $\nu_0 \in W(A_1)'$. Assume $\nu_0 \notin W(A_1)'$. Then $\nu_0(A_1) = 1$. Hence, by the definition of ϕ, $\mu_0 \in A_1^{-IR\mathscr{L}(1)}$. Hence, since $W(A_1)'$ is a neighbourhood of μ_0, $A_1 \cap W(A_1)' \neq \phi$. Hence $A_1 \cap A_1' \neq \phi$. Since this statement is false, the assumption is wrong. Consequently $\nu_0 \in W(A_1)'$. Consequently $W(A_1)'$ is a $tW(\mathscr{L}_2)$-neighbourhood of ν_0.

(β) Show $\phi(W(A_1)') \subset W(A')$. Consider any element of $\phi(W(A_1)')$, $\phi(\nu)$. Then $\nu(A_1') = 1$. Hence, since $\nu \in IR(\mathscr{L}_2)$, there exists an element of $\mathscr{L}_2$, B, such that $B \subset A_1'$ and $\nu(B) = 1$. Consider any such B. Then, by the definition of ϕ, $\phi(\nu) \in \bar{B}^{IR(\mathscr{L}_1)}$. Hence, since $B \subset A_1'$, $\phi(\nu) \in \overline{A_1'}^{IR(\mathscr{L}_1)} \subset \overline{W(A_1)'}$. Hence, since $\overline{W(A_1)'} \subset W(A)'$, $\phi(\nu) \in W(A)'$. Hence $\phi(W(A_1)') \subset W(A)'$.

(γ) Consequently $W(A_1)'$ is a $tW(\mathscr{L}_2)$-neighbourhood of ν_0 and $\phi(W(A_1)') \subset W(A)'$. Hence ϕ is continuous at ν_0.

Hence ϕ is continuous.

(ii) Assume $\mathscr{L}_2 = \mathscr{L}(\mathscr{R}(t\,\mathscr{L}_1))$.

Consider $\overset{\circ}{\overline{W(A_1)'}}$. Note $\overset{\circ}{\overline{W(A_1)'}}$ is regular open in $IR(\mathscr{L}_1)$. Further, consider $X \cap \overset{\circ}{\overline{W(A_1)'}}$. Then, since X is dense in $IR(\mathscr{L}_1)$, (by Proposition 2.4, (i)), X Set $X \cap \overset{\circ}{\overline{W(A_1)'}}$ is regular open in X. $O \cap \overset{\circ}{\overline{W(A_1)'}} = O$. Note O' is regular closed in X. Hence $O' \in \mathscr{L}_2$. Hence $W(O')'$-which is equal to $W(O)$-is $tW(\mathscr{L}_2)$-open.

(α) Show $W(O)$ is a $tW(\mathscr{L}_2)$-neighbourhood of ν_0. Since $W(O)$ is $tW(\mathscr{L}_2)$-open, it suffices to show $\nu_0 \in W(O)$. Assume $\nu_0 \notin W(O)$. Then $\nu_0(O') = 1$. Hence, by the definition of ϕ, $\mu_0 \in \bar{O}'^{IR(\mathscr{L}_1)}$. Hence, since $\overset{\circ}{\overline{W(A_1)'}}$ is a neighbourhood of of μ_0, $O' \cap \overset{\circ}{\overline{W(A_1)'}} \neq \phi$. Hence $(O' \cap X) \cap \overset{\circ}{\overline{W(A_1)'}} \neq \phi$. Hence, since $X \cap \overset{\circ}{\overline{W(A_1)'}} = O$, $O' \cap O \neq \phi$. Since this statement is false, the assumption is wrong. Consequently $\nu_0 \in W(O)$. Consequently $W(O)$ is a $t\,W(\mathscr{L}_2)$-neighbourhood of ν_0.

(β) Show $\phi(W(O)) \subset W(A)'$. Consider any element of $\phi(W(O))$, $\phi(\nu)$. Then $\nu(O) = 1$. Further, note since O is open in X, $\bar{O}^{t\mathscr{L}_1}$ is regular closed (in X). Hence $\bar{O}^{t\mathscr{L}_1} \in \mathscr{L}_2$. Consequently $\nu(\bar{O}^{t\mathscr{L}_1}) = 1$. Then, by the definition of ϕ,

$$\phi(\nu) \in \overline{\bar{O}^{t\mathscr{L}_1}}^{IR(\mathscr{L}_1)}. \text{ Note } \bar{O}^{t\mathscr{L}_1} \subset \bar{O}^{IR(\mathscr{L}_1)}. \text{ Then } \overline{\bar{O}^{t\mathscr{L}_1}}^{IR(\mathscr{L}_1)}$$

$$\subset \bar{O}^{IR(\mathscr{L}_1)} \subset \overline{\overset{\circ}{\overline{W(A_1)'}}}^{IR(\mathscr{L}_1)}, \text{ since } O \subset \overset{\circ}{\overline{W(A_1)'}}$$

$= \overline{W(A_1)'}$, since $\overline{W(A_1)'}$ is regular closed.

$\subset W(A)'$. Consequently $\phi(\nu) \in W(A)'$. Hence $\phi(W(O))(W \subset (A)'$.

(γ) Consequently $W(O)$ is a $tW(\mathscr{L}_2)$-neighbourhood of ν_0 and $\phi(W(O)) \subset W(A)'$. Hence ϕ is continuous at ν_0.

Hence ϕ is continuous.

(iii) Assume $\mathscr{L}_2$ = the Boolean algebra $\mathscr{R}(t\mathscr{L}_1)$.

(α) Consider $A_1'^{\,t\mathscr{L}_1}$ and set $\overline{A_1'}^{\,t\mathscr{L}_1} = B$. Set $B^* = B_1$. Note $B_1^* = (B^*)^* = \bar{B} = A_1'^{\,t\mathscr{L}_1}$. Then, since $W(A_1)'$ is a neighbourhood of μ_0, by the observation preceding the definition of ϕ, $\nu(B_1^*) = 1$. Now, consider $W(B_1^*)$. Then, by (Proposition 2.1 (α)), $W(B_1^*) = W(B_1)'$. Hence $W(B_1^*) \in t(W(\mathscr{L}_2))'$. Further, note $\nu_0 \in W(B_1^*)$. Consequently $W(B_1^*)$ is a $tW(\mathscr{L}_2)$-neighbourhood of ν_0.

(β) Show $\phi(W(B_1^*)) \subset W(A)'$. Consider any element of $\phi(W(B_1^*))$, $\phi(\nu)$. Then $\nu(B_1^*) = 1$. Consequently

$\nu(\overline{A_1'}^{\,t\mathscr{L}_1}) = 1$. Hence, by the definition of ϕ,

$$\phi(\nu) \in \overline{\overline{A_1'}^{\,t\mathscr{L}_1}}^{\,IR(\mathscr{L}_1)}. \quad \text{Note } \overline{A_1'}^{\,t\mathscr{L}_1} \subset \overline{A_1'}^{\,IR(\mathscr{L}_1)}. \quad \text{Then}$$

$$\overline{\overline{A_1'}^{\,t\mathscr{L}_1}}^{\,IR(\mathscr{L}_1)} \subset \overline{A_1'}^{\,IR(\mathscr{L}_1)} \subset \overline{W(A_1)'} \subset W(A)'.$$

Consequently $\phi(\nu) \in W(A)'$.

Hence $\phi(W(B_1^*)) \subset W(A)'$.

(γ) Consequently $W(B_1^*)$ is a $tW(\mathscr{L}_2)$-neighbourhood of ν_0 and $\phi(W(B_1^*)) \subset W(A)'$. Hence ϕ is continuous at ν_0.

Hence ϕ is continuous.

Proposition 2.7—If $\mathscr{L}_2$ is separating and disjunctive, then $\phi(X) = X$, pointwise.

PROOF : Assume $\mathscr{L}_2$ is separating anh disjunctive. Denote the general element of X by x, the element of $IR(\mathscr{L}_1)$ which is concentrated at x by μ_x, and the element of $IR(\mathscr{L}_2)$ which is concentrated at x by μ_x.

(α) Show $\phi(\nu_x) = \mu_x$. Consider any element of $\mathscr{L}_2$, B, such that $\nu_x(B) = 1$. Then $x \in B$. Consequently $\mu_x \in \bar{B}^{IR(\mathscr{L}_1)}$. Then, by the definition of ϕ, $\phi(\nu_x) = \mu_x$.

(β) Since $\mathscr{L}_2$ is separating and disjunctive, X is (densely) embeddable in $IR(\mathscr{L}_2)$. Accordingly, embed X into $IR(\mathscr{L}_2)$. Then, because of (α), $\phi(X) = X$, pointwise.

§3. We continue the study of the function ϕ for the case $t\mathscr{L}_1 \subset \mathscr{L}_2$.

(It is to be remembered that for the definition of ϕ, $\mathscr{L}_1$ is assumed to be normal, separating, and disjunctive.)

Proposition 3.1—(i) ϕ is onto.

(ii) If $\mathscr{L}_1$ semiseparates $\mathscr{L}'_2$, then ϕ is identical with the function which maps the general element of $IR(\mathscr{L}_2)$ to its restriction to $\mathscr{H}(\mathscr{L}_1)$.

Proof : (i) Consider any element of $IR(\mathscr{L}_1)$, μ, and show there exists an element of $IR(\mathscr{L}_2)$, ν, such that $\phi(\nu) = \mu$. Since $\mathscr{L}_1 \subset \mathscr{L}_2$, there exists an element of $IR(\mathscr{L}_2)$, ν, such that $\nu \mid_{\mathscr{H}(\mathscr{L}_1)} = \mu$. Consider any such ν. Show $\phi(\nu) = \mu$. Consider any element of $\mathscr{L}_2, B$, such that $\nu(B) = 1$ and show $\mu \in \bar{B}^{IR(\mathscr{L}_1)}$. Assume $\mu \notin \bar{B}^{IR(\mathscr{L}_1)}$. Then there exists a basic neighbourhood of μ, $W(A)'$, such that $B \cap W(A)' = \phi$. Consider any such $W(A)'$. Then $B \cap A' = \phi$. Hence $B \subset A$. Hence, since $\nu(B) = 1$ and $A \in \mathscr{L}_2$, $\nu(A) = 1$. Then, since $\nu \mid_{\mathscr{H}(\mathscr{L}_1)} = \mu$, $\mu(A) = 1$. Since $\mu \in W(A)'$, $\mu(A) = 0$. Thus a contradiction had been reached. Consequently $\mu \in \bar{B}^{IR(t\mathscr{L}_1)}$. Hence, by the definition of ϕ, $\phi(\nu) = \mu$. Thus ϕ is onto.

(ii) Assume $\mathscr{L}_1$ semiseparates $\mathscr{L}_2$.

Consider any element of $IR(\mathscr{L}_2)$, ν, and set $\phi(\nu) = \mu$. Show $\nu \mid_{\mathscr{H}(\mathscr{L}_1)} = \mu$.

Proof : Since $\nu \in IR(\mathscr{L}_2)$ and $\mathscr{L}_1$ semiseparates $\mathscr{L}_2$, $\nu \mid_{\mathscr{H}({}_1\mathscr{L})} \in IR(\mathscr{L}_1)$. Hence, to show $\nu \mid_{\mathscr{H}(\mathscr{L}_1)} = \mu$, it suffices to show $\nu \mid_{\mathscr{H}(\mathscr{L}_1)} \leqq \mu$ on $\mathscr{L}_1$. Accordingly, consider any element of $\mathscr{L}_1$, A, such that $\nu(A) = 1$ and show $\mu(A) = 1$. Assume $\mu(A) = 0$. Since $\nu(A) = 1$, by the definition of ϕ, $\mu \in \bar{A}^{IR(\mathscr{L}_1)}$ and, since $\mu(A) = 0$, $\mu \in W(A)'$. Hence $A \cap W(A)' \neq \phi$. Hence $A \cap A' \neq \phi$. Since this statment is false, the assumption is wrong. Consequently $\mu(A) = 1$. Thus $\nu \mid_{\mathscr{H}(\mathscr{L}_1)} \leqq \mu$ on $\mathscr{L}_1$. Consequently $\nu \mid_{\mathscr{H}(\mathscr{L}_1)} = \mu$.

Note : Consider the function ψ which maps the general element of $IR(\mathscr{L}_2)$ to its restriction to $\mathscr{H}(\mathscr{L}_1)$. The following statement is true : If $\mathscr{L}_2$ is is disjunctive and $\psi(IR(\mathscr{L}_2)) \subset IR(\mathscr{L}_1)$, then $\mathscr{L}_1$ semiseparates $\mathscr{L}_2$. (Proof omitted.)

Remark : The normality of $\mathscr{L}_1$ is used in the proof.

Applications—(1) Consider any topological space X such that X is $T_{3\frac{1}{2}}$ and let $\mathscr{L}_1 = \mathscr{Z}$ and $\mathscr{L}_2 = \mathscr{F}$. Note $\mathscr{L}_1$ is normal, separating, and disjunctive, $t\mathscr{L}_1 \subset \mathscr{L}_2$, and $\mathscr{L}_2$ is separating and disjunctive. Hence, by [(Proposition 2.6, (i)), Proposition 2.7,

and (Proposition 3.1, (i))], $IR(\mathscr{L}_1)$ is the continuous image of $IR(\mathscr{L}_2)$ under a function which leaves X fixed, pointwise, that is, $IR(\mathscr{Z})$ is the continuous image of $IR(\mathscr{F})$ under a function which leaves X fixed, pointwise, in other (standard) symbols, βX Gillman and Jerison[9] is the continuous image of ωX Wallman[16] under a function which leaves X fixed, pointwise.

(2) Consider any topological space X such that X is T_1 and 0-dimensional and let $\mathscr{L}_1 = \mathscr{C}$ and $\mathscr{L}_2 = \mathscr{F}$. Note $\mathscr{L}_1$ is normal, separating, and disjunctive, $t\ \mathscr{L}_1 \subset \mathscr{L}_2$, and $\mathscr{L}_2$ is separatin gand disjunctive. Hence, by [(Proposition 2.6, (i)), Proposition 2.7, and (Proposition 3.1, (i))], $IR(\mathscr{L}_1)$ is the continuous image of $IR(\mathscr{L}_2)$ under a function which leaves X fixed, pointwise, that is $IR(\mathscr{C})$ is the continuous image of $IR(\mathscr{F})$ under a function which leaves X fixed, pointwise, in other (standard) symbols, $\beta_0 X$ (Banaschewski[4]) is the continuous image of ωX under a function which leaves X fixed, pointwise.

§3. Consider any topological space X such that X is $T_{3\frac{1}{2}}$ and countably bounded and let $\mathscr{L}_1 = \mathscr{Z}$ and $\mathscr{L}_2 = \mathscr{F}$. Note $\mathscr{L}_1$ is normal, separating, and disjunctive. Moreover, since X is countably bounded, $\mathscr{L}_1$ semiseparates $\mathscr{L}_2$ (Nöbeling[14]). Then, by (Proposition 3.1, (ii)), ϕ is identical with the function which maps the general element of $IR(\mathscr{F})$ to its restriction to $\mathscr{A}(\mathscr{L})$.

Theorem 3.2—If $\mathscr{L}_1$ separates $\mathscr{L}_2$, then ϕ is a homeomorphim between $IR(\mathscr{L}_2)$ and $IR(\mathscr{L}_1)$.

Moreover, if $\mathscr{L}_2$ is separating and disjunctive, then $\phi(X) = X$, pointwise.

PROOF : Assume $\mathscr{L}_1$ separates $\mathscr{L}_2$. Then ϕ is a one-to-one correspondence between $IR(\mathscr{L}_2)$ and $IR(\mathscr{L}_1)$. (See Proposition 3.1). Hence, since ϕ is continuous (see Proposition 2.6, (i)) and $IR(\mathscr{L}_1)$ is T_2, ϕ is a homeomorphism between $IR(\mathscr{L}_2)$ and $IR(\mathscr{L}_1)$.

Further, assume $\mathscr{L}_2$ is separating and disjunctive. Then $\phi(X) = X$, pointwise. (See Proposition 2.7.)

Applications—(1) Consider any topological space X such that X is T_1 and normal and let $\mathscr{L}_1 = \mathscr{Z}$ and $\mathscr{L}_2 = \mathscr{F}$. Then, by Theorem 3.2, $IR(\mathscr{F})$ and $IR(\mathscr{Z})$ are homeomorphic, in other symbols, ωX and βX are homeomorphic.

(2) Consider any topological space X such that X is T_1 and 0-dimensional and ultranormal (i. e., $\mathscr{C}$ separates $\mathscr{F}$) and let $\mathscr{L}_1 = \mathscr{C}$ and $\mathscr{L}_2 = \mathscr{F}$. Then, by Theorem 3.2, $IR\ \mathscr{F}$ and $IR(\mathscr{C})$ are homeomorphic, in other symbols, ωX and $\beta_0 X$ are homeomorphic.

Observation : Since X is T_1 and ultranormal, X is T_1 and normal, Hence ωX and βX are homeomorphic. (See the preceding application.) Consequently any two of the topological spaces, ωX, βX, $\beta_0 X$ are homeomorphic.

Theorem 3.3--(i) If $\mathscr{L}_1$ is countably paracompact, then ϕ $(IR\ (\sigma, \mathscr{L}_2)) \subset IR\ (\sigma, \mathscr{L}_1)$.

(ii) If $\mathscr{L}_1$ semiseparates $\mathscr{L}_2$ and $\mathscr{L}_2$ is $\mathscr{L}_1$-countably bounded, or $\mathscr{L}_2$ is $\mathscr{L}_1$-countably paracompact, then $IR\ (\sigma, \mathscr{L}_1) \subset \phi\ (IR\ (\sigma, \mathscr{L}_2))$.

PROOF : (i) Assume $\mathscr{L}_1$ is countably paracompact. Consider any element of $\phi\ (IR\ (\sigma, \mathscr{L}_2))$, $\phi\ (\nu)$. Set $\phi\ (\nu) = \mu$. Note to show $\mu \in IR\ (\sigma, \mathscr{L}_1)$, since $\mu \in IR\ (\mathscr{L}_1)$, it suffices to show for every sequence in $\mathscr{L}_1$, $<A_n>$, if $<A_n>$ is decreasing and $\lim_n A_n = \phi$, then $\lim_n \mu\ (A_n) = 0$. Accordingly, consider any sequence in $\mathscr{L}_1$, $< A_n >$, such that ... and show $\lim_n \mu < A_n > = 0$. Assume $\lim_n \mu\ (A_n) \neq 0$. Since ($< A_n >$ is in $\mathscr{L}_1$ and $< A_n >$ is decreasing and $\lim_n A_n = \phi$) and $\mathscr{L}_1$ is countably paracompact, there exists a sequence in $\mathscr{L}_1$, $< C_n >$, such that for every n, $A_n \subset C'_n$ and $<C'_n>$ is decreasing and $\lim_n C'_n = \phi$. Consider any such $<C_n>$. Since $<A_n>$ and $<C_n>$ are in $\mathscr{L}_1$ and $\mathscr{L}_1 \subset \mathscr{L}_2$, $<A_n>$ and $<C_n>$ are in $\mathscr{L}_2$. Then, since $\nu \in IR\ (\sigma, \mathscr{L}_2,)$ $\lim_n \nu\ (C'_n)=0$. Hence there exists a value of n, m, such that $\nu\ (C'_m) = 0$. Consider any such m. Then $\nu\ (C_m) = 1$. Hence, by the definition of ϕ, $\mu \in \mathcal{C}_m^{IR(\mathscr{L}_1)}$. Since $\lim_n \mu\ (A_n) \neq 0$ $\mu\ (A_m) = 1$. Hence, since $\mathscr{A}_m \subset C'_m$, $\mu\ (C'_m = 1$. Hence $\mu \in W\ (C'_m)$. Hence, since $\mu \in \mathcal{C}_m^{IR(\mathscr{L}_1)}$, $C_m \cap W\ (C'_m) \neq \phi$. Hence $C_m \cap C'_m \neq \phi$. Since this statement is false, the assumption is wrong. Consequently $\lim_n \mu\ (A_n) = 0$. Consequently $\mu \in IR\ (\sigma, \mathscr{L}_1)$. Thus $\phi\ (IR\ (\sigma, \mathscr{L}_2)) \subset IR\ (\sigma, \mathscr{L}_1)$.

(ii) Assume $\mathscr{L}_1$ semiseparates $\mathscr{L}_2$ and $\mathscr{L}_2$ is $\mathscr{L}_1$-countably bounded, or $\mathscr{L}_2$ is $\mathscr{L}_1$-countably paracompact. Since $\mathscr{L}_1$ semi-separates $\mathscr{L}_2$, ϕ coincides with the function which maps the general element of $IR\ (\mathscr{L}_2)$ to its restriction to $\mathscr{A}\ (\mathscr{L}_1)$. Then the proof of the statement $IR\ (\sigma, \mathscr{L}_1) \subset \phi\ (IR\ (\sigma, \mathscr{L}_2)$ is well-known.

Note : The following statement is true : If $\mathscr{L}_2$ is $\mathscr{L}_1$-countably bounded, then $IR\ (\sigma, \mathscr{L}_1) \subset \phi\ (IR\ (\sigma, \mathscr{L}_2))$.

PROOF : Assume $\mathscr{L}_2$ is $\mathscr{L}_1$-countably bounded. Since $\mathscr{L}_1$ is disjunctive, $IR\ (\sigma, \mathscr{L}_1) \neq \phi$. Consider any element of $IR\ (\sigma, \mathscr{L}_1)$, μ. Then, since ϕ is onto, by (Proposition 3.1, (i)), there exists an element of $IR\ (\mathscr{L}_2)$, ν, such that $\phi\ (\nu) = \mu$. Consider any such ν. Note to show $\nu \in IR\ (\sigma, \mathscr{L}_2)$, since $\nu \in IR\ (\mathscr{L}_2)$, it suffices to show for every sequence in $\mathscr{L}_2$, $<B_n>$, if $<B_n>$ is decreasing and $\lim_n B_n = \phi$, then $\lim_n \nu\ (B_n) = 0$. Accordingly, consider any sequence in $\mathscr{L}_2$, $<B_n>$, such that...and show $\lim_n \nu$

$(B_n) = 0$. Assume $\lim_n \nu(B_n) \neq 0$. Since ($<B_n>$ is in $\mathscr{L}_2$ and $<B_n>$ is decreasing and $\lim_n B_n = \phi$) and $\mathscr{L}_2$ is $\mathscr{L}_1$-countably bounded, there exists a sequence in $\mathscr{L}_1$, $<A_n>$, such that for every n, $B_n \subset A_n$ and $<A_n>$ is decreasing and $\lim_n A_n = \phi$. Consider any such $<A_n>$. Note for every n, since $\lim_n \nu(B_n) \neq 0$ and $\nu \in I(\mathscr{L}_2)$ and $<B_n>$ is decreasing $\nu(B_n) = 1$; then, by the definition of ϕ, $\mu = \phi(\nu) \in \bar{B}^{IR(\mathscr{L}_1)}$; further, note $\bar{B}_n^{IR(\mathscr{L}_1)} \subset A_n^{IR(\mathscr{L}_1)} \subset W(A_n)$, since $B_n \subset A_n$ and $A_n \in \mathscr{L}_1$; consequently $\mu \in W(A_n)$; hence $\mu(A_n) = 1$. Hence $\lim_n \mu(A_n) = 1$. Further, note, since ($<A_n>$ isin $\mathscr{L}_1$ and $<A_n>$ is decreasing and $\lim_n A_n = \phi$) and $\mu \in IR(\sigma, \mathscr{L}_1)$, $\lim_n \mu(A_n) = 0$. Thus a contradiction has been reached. Consequently $\lim_n \nu(B_n) = 0$. Consequently $\nu \in IR(\sigma, \mathscr{L}_2)$. Hence $IR(\sigma, \mathscr{L}_1) \subset \phi(IR(\sigma, \mathscr{L}_2))$.

Colollary 3.4—If $\mathscr{L}_1$ is countably paracompact and $\mathscr{L}_2 = t\mathscr{L}_1$, then $\mathscr{L}_1$ is replete $\Rightarrow$ $\mathscr{L}_2$ is replete.

Proof: Assume $\mathscr{L}_1$ is countably paracompact and $\mathscr{L}_2 = t\,\mathscr{L}_1$, Further, assume $\mathscr{L}_1$ is replete. Note to show $\mathscr{L}_2$ is replete, according to the definition of repleteness, it suffices to show for every element of $IR(\mathscr{L}_2)$, ν, if $\nu \in IR(\sigma, \mathscr{L}_2)$, then $S(\nu) \neq \phi$. Accordingly, assume $IR(\sigma, \mathscr{L}_2) \neq \phi$ and consider any element of $IR(\sigma, \mathscr{L}_2)$ ν. Consequently $\phi(\nu) \in IR(\sigma, \mathscr{L}_1)$. Hence, since $\mathscr{L}_1$ is replete, $S(\phi(\nu)) \neq \phi$. Hence there exists an element of X, x, such that $\phi(\nu) = \mu_x$ (and, since $\mathscr{L}_1$ is separating, x is unique). Now, consider the element of $I(\mathscr{L}_2)$ which is concentrated at x and denote it by ν_x. Show $\nu = \nu_x$. Since $\nu \in IR(\mathscr{L}_2)$, it suffices to show $\nu \leqq \nu_x$ on $\mathscr{L}_2$. Accordingly, consider any element of $\mathscr{L}_2$, B, such that $\nu(B) = 1$. Then, by the definition of ϕ, $\mu_x = \phi(\nu) \in \bar{B}^{IR(\mathscr{L}_1)}$. Note $X \cap \bar{B}^{IR(\mathscr{L}_1)} = \bar{B}^{t\mathscr{L}_1}$. Moreover, since $B \in \mathscr{L}_2$ and $\mathscr{L}_2 = t\,\mathscr{L}_1$, $B \in t\,\mathscr{L}_1$. Consequently $x \in B$. Hence $\nu_x(B) = 1$. Consequently $\nu \leqq \nu_x$ on $\mathscr{L}_2$. Consequently $\nu = \nu_x$. Hence $S(\nu) \neq \phi$. Consequently $\mathscr{L}_2$ is replete.

Note: The statement "If $\mathscr{L}_2$ is $\mathscr{L}_1$-countably bounded, of $\mathscr{L}_2$ is $\mathscr{L}_1$-countably paracompact, then $\mathscr{L}_2$ is replete $\Rightarrow$ $\mathscr{L}_1$ is replete" is true regardless of whether $\mathscr{L}_1$ is normal, separating, or disjunctive. For a proof of this statement and related applications, see Bachman and Stratigos[2] (pp. 543, 544).

Applications—(1) Consider any topological space X such that X is $T_{3\frac{1}{2}}$ and let $\mathscr{L}_1 = \mathscr{Z}$ and $\mathscr{L}_2 = \mathscr{F}$. Then, by Corollary 3.4, $\mathscr{Z}$ is replete $\Rightarrow$ $\mathscr{F}$ is replete; otherwise stated : X is realcompact $\Rightarrow$ X is α-complete. (see Gillman and Jerison[9] and Dykes[6] for the terminology used.)

* Since $\mathscr{L}_1$ is countably paracompact, by (Theorem 3.3, (i)), $\phi(IR(\sigma, \mathscr{L}_1)) \subset IR(\sigma, \mathscr{L}_1)$.

(2) Consider any topological space X such that X is T_1 and 0-dimensional and let $\mathscr{L}_1 = \mathscr{C}$ and $\mathscr{L}'_2 = \mathscr{F}$. Then by Corollary 3.4, $\mathscr{C}$ is replete $\Rightarrow$ $\mathscr{F}$ is replete; otherwise stated : X is N-compact[11] $\Rightarrow$ X is α-complete.

Remark : The results described above were obtained earlier under a different setting[2] (p. 543).

§ 4. We continue the study of the function ϕ for the case $\mathscr{L}_2 = \mathscr{L}\ (\mathscr{R}\ (t\mathscr{L}_1))$. (It is to be remembered that for the definition of ϕ, $\mathscr{L}_1$ is assumed to be normal, separating and disjunctive.)

Proposition 4.1—ϕ is onto.

PROOF : Consider any element of $IR\ (\mathscr{L}_1)$, μ, and show there exists an element of $IR\ (\mathscr{L}_2)$, ν, such that $\phi\ (\nu) = \mu$.

Consider the set whose general element is an element of $W(\mathscr{L}_1)'$, $W(A)'$, such that $\mu \in W(A)'$. Denote this set by $\{O_\lambda;\ \lambda \in \Lambda\}$. Note for every λ, $\bar{O}_\lambda^{IR(\mathscr{L}_1)}$ is $tW(\mathscr{L}_1)$-regular closed; hence, by (Proposition 2.4 (i)), $X \cap \bar{O}_\lambda^{IR\ (\mathscr{L}_1)}$ is $t(\mathscr{L}_1)$ regular closed; hence $X \cap \bar{O}_\lambda^{IR(\mathscr{L}_1)} \in \mathscr{L}_2$. Consider $\{X \cap \bar{O}_\lambda^{IR(\mathscr{L}_1)}\ ; \lambda \in \Lambda\}$. Show it is a base for an $\mathscr{L}_2$-filter. Note for every λ, $X \cap \bar{O}_\lambda^{IR(\mathscr{L}_1)} \supset X \cap O_\lambda \neq \phi$, since $\mu \in O_\lambda \in W(\mathscr{L}_1)'$ and X is dense in $IR\ (\mathscr{L}_1)$. Further, note for any two values of λ, λ_1, λ_2, $(X \cap \bar{O}_{\lambda_1}^{IR(\mathscr{L}_1)}) \cap (X \cap \bar{O}_{\lambda_1}^{IR(\mathscr{L}_2)}) = X \cap (\bar{O}_{\lambda_1}^{IR(\mathscr{L}_2)} \cap \bar{O}_{\lambda_1}^{IR(\mathscr{L}_1)})$ $\supset X \cap \overline{O_{\lambda_1} \cap O_{\lambda_2}}^{IR(\mathscr{L}_1)}$ and $\mu \in O_{\lambda_1} \cap O_{\lambda_2} \in W(A)'$. Consequently $\{X \cap \bar{O}_\lambda^{IR(\mathscr{L}_1)}\ ; \lambda \in \Lambda\}$ is a base for an $\mathscr{L}_2$-filter. Therefore there exists an element of $IR\ (\mathscr{L}_2)$, ν, such that for every λ, $\nu\ (X \cap \bar{O}^{IR(\mathscr{L}_1)}) = 1$. Consider any such ν. Then $\{\phi\ (\nu)\} = \cap\ \{\bar{B}^{IR(\mathscr{L}_1)} / B \in \mathscr{L}_2$ and $\nu\ (B) = 1\}$, by the definition of ϕ, $\subset \cap\ \{\bar{O}_\lambda^{IR(\mathscr{L}_1)}\ ; \lambda \in \Lambda\}$, since for every λ, $\nu\ (X \cap \bar{O}_\lambda^{IR(-1)}) = 1, = \{\mu\}$, since $\mu \in \cap\ \{\bar{O}_\lambda^{IR(\mathscr{L}_1)}\ ; \lambda \in \Lambda\}$ and $IR\ (\mathscr{L}_1)$ is regular and T_2.

Hence $\phi\ (\nu) = \mu$.

Hence ϕ is onto.

Application—Consider any topological space X such that X is $T_{3\frac{1}{2}}$ and let $\mathscr{L}_1 = \mathscr{Z}$ and $\mathscr{L}_2 = \mathscr{F}_R$.

The following fact appears in Walker[15] (p. 10): Every T_2-compactification of X is the continuous image of βX under a mapping which leaves X fixed, pointwise.

(α) Note since X is $T_{3\frac{1}{2}}$, $\mathscr{F}_R$ is separating and disjunctive. Hence $IR\ (\mathscr{F}_R)$ is a compactification of X. Further, assume $\mathscr{F}_R$ is normal. Then, since $\mathscr{F}_R$ is dis-

junctive, $IR(\mathscr{F}_R)$ is T_2. Consequently $IR(\mathscr{F}_R)$ is a T_2-compactification of X. Then, by the fact mentioned above, $IR(\mathscr{F}_R)$ is the continuous image of βX under a mapping which leaves X fixed, pointwise.

(β) By Proposition 4.1, [Proposition 2.6, (ii)], and Proposition 2.7, $IR(\mathscr{Z})$ ($=\beta X$) is the continuous image of $IR(\mathscr{F}_R)$ under a mapping which leaves X fixed, pointwise.

(γ) Consequently $IR(\mathscr{F}_R)$ and βX are homeomorphic. (The proof of this statement involves a standard "denseness" argument.)

(δ) The following statement is also true : If $IR(\mathscr{F}_R)$ and βX are homeomorphic, then $IR(\mathscr{F}_R)$ is T_2 and, since $\mathscr{F}_R$ is disjunctive, $\mathscr{F}_R$ is normal.

Remark : This result was also established by Zaicev[17], under a different setting. (Zaicev calls such spaces in which $\mathscr{F}_B$ is normal quasi-normal and uses the notation $IR(\mathscr{F}_R) = w_\pi(X)$.)

Proposition 4.2—If $\mathscr{L}_1$ separates $t\mathscr{L}_1$, then ϕ is one-to-one.

PROOF : Assume $\mathscr{L}_1$ separates $t\mathscr{L}_1$. Consider any two elements of $IR(\mathscr{L}_2)$, ν_1, ν_2, such that $\nu_1 \neq \nu_2$ and show $\phi(\nu_1) \neq \phi(\nu_2)$. Note to show $\phi(\nu_1) \neq \phi(\nu_2)$, according to the definition of ϕ, it suffices to show there exist two elements of $\mathscr{L}_2$, B_1, B_2, such that $\nu_1(B_1) = 1$ and $\nu_2(B_2) = 1$ and $\bar{B}_1^{IR(\mathscr{L}_1)} \cap \bar{B}_2^{IR(\mathscr{L}_2)} = \phi$. Since $\nu_1, \nu_2 \in IR(\mathscr{L}_2)$ and $\nu_1 \neq \nu_2$, there exists two elements of $\mathscr{L}_2$, B_1, B_2, such that $\nu_i(B_j) = \delta_{ij}$ $(i, j = 1, 2)$ and $B_1 \cap B_2 = \phi$. Consider any such B_1, B_2. Then, since $\mathscr{L}_1$ separates $t\mathscr{L}_1$, there exist two elements of $\mathscr{L}_1$, A_1, A_2, such that $B_1 \subset A_1$ and $B_2 \subset A_{21}$ and $A_1 \cap A_2 = \phi$. Consider any such A_1, A_2. Then $\bar{B}_1^{IR(\mathscr{L}_1)} \cap \bar{B}_2^{IR(\mathscr{L}_1)} \subset \bar{A}_1^{IR(\mathscr{L}_1)} \cap \bar{A}_2^{IR(\mathscr{L}_1)} \subset W(A_1) \cap W(A_2) = W(A_1 \cap A_2) = W(\phi) = \phi$. Hence $\bar{B}_1^{IR(\mathscr{L}_1)} \cap \bar{B}_2^{IR(\mathscr{L}_1)} = \phi$. Consequently $\phi(\nu_1) \neq \phi(\nu_2)$. Hence ϕ is one-to-one.

Theorem 4.3—If $\mathscr{L}_1$ separates $t\mathscr{L}_1$, then ϕ is a homeomorphism between $IR(\mathscr{L}_2)$ and $IR(\mathscr{L}_1)$.

Moreover, if $\mathscr{L}_2$ is separating and disjunctive, then $\phi(X) = X$, pointwise.

PROOF : Assume $\mathscr{L}_1$ separates $t\mathscr{L}_1$. Note ϕ is onto (Proposition 4.1). Further, note, since $\mathscr{L}_1$ separates $t\mathscr{L}_1$, ϕ is one-to-one (Proposition 4.2). Consequently ϕ is a one-to-one correspondence between $IR(\mathscr{L}_2)$ and $IR(\mathscr{L}_1)$. Hence, since ϕ is continuous (see Proposition 2.6. (ii)) and $IR(\mathscr{L}_1)$ is T_2, ϕ is a homeomorphism between $IR(\mathscr{L}_2)$ and $IR(\mathscr{L}_1)$.

Further, assume $\mathscr{L}_2$ is separating and disjunctive. Then $\phi(X) = X$, pointwise. (See Proposition 2.7).

Observation : Since $IR(\mathscr{L}_1)$ is normal, $IR(\mathscr{L}_2)$ is normal. Consequently $\mathscr{L}_2$ is normal.

Applications —(1) Consider any topological space X such that X is T_1 and normal. Note X is $T_{3\frac{1}{2}}$. Hence $t\mathscr{Z} = \mathscr{F}$.

Let

$$\mathscr{L}_1 = \mathscr{Z} \text{ and } \mathscr{L}_2 = \mathscr{L}(R(t\mathscr{L}_1)) \equiv \mathscr{F}_R.$$

Note, since X is normal, $\mathscr{L}_1$ separates $t\,\mathscr{L}_1$. Then, by Theorem 4.3, $IR(\mathscr{L}_2)$ and $IR(\mathscr{L}_1)$ are homeomorphic, in other symbols, $\omega_\pi X$ and βX are homeomorphic.

Further, note $\mathscr{L}_2$ is separating and disjunctive. Then, by Proposition 2.7, $\phi(X) = X$, pointwise.

(2) Consider any topological space X such that X is T_1 and 0-dimensional and ultranormal (i. e., $\mathscr{C}$ separates $\mathscr{F}$). Since X is T_1 and 0-dimensional, $t\mathscr{C} = \mathscr{F}$.

Let $\mathscr{L}_1 = \mathscr{C}$ and $\mathscr{L}_2 = \mathscr{L}(\mathscr{R}(t\mathscr{L}_1))$. (Note $\mathscr{L}(\mathscr{R}(t\mathscr{C})) = \mathscr{F}_R$).

Note, since X is ultranormal, $\mathscr{L}_1$ separates $t\mathscr{L}_1$. Then, by Theorem 4.3, $IR(\mathscr{L}_2)$ and $IR(\mathscr{L}_1)$ are homemorphic, in other symbols, $\omega_\pi X$ and $\beta_0 X$ are homeomorphic.

Observation : Under the above conditions, any two of the topological spaces ωX, βX, $\beta_0 X$, $\omega_\pi X$ are homeomorphic (see Application 2 after Theorem 3.2.)

The following definition will play an important role in the sequel.

Definition—$t\,\mathscr{L}_1$ is $\mathscr{L}_1$-weakly countably bounded iff $\mathscr{L}_2\,(= \mathscr{L}(\mathscr{R}(t\,\mathscr{L}_1)))$ is $\mathscr{L}_1$-countably bounded.

Example—Consider any topological space X such that X is $T_{3\frac{1}{2}}$ and let $\mathscr{L}_1 = \mathscr{Z}$. Note $\mathscr{L}_2 = \mathscr{F}_R$. Then, according to the preceding definition, $\mathscr{F}$ is $\mathscr{Z}$-weakly countably bounded iff $\mathscr{F}_R$ is $\mathscr{Z}$-countably bounded. X is said to be weakly countably bounded iff $\mathscr{F}$ is $\mathscr{Z}$-weakly countably bounded.

Theorem 4.4—(i) If $\mathscr{L}_1$ is countably paracompact, then $\phi\,(IR\,(\sigma, \mathscr{L}_2)) \subset IR\,(\sigma, \mathscr{L}_1))$.

(ii) If $t\,\mathscr{L}_1$ is $\mathscr{L}_1$-weakly countably bounded, then $IR\,(\sigma, \mathscr{L}_1) \subset \phi\,(IR\,(\sigma, \mathscr{L}_2))$.

Proof : (i) Assume $\mathscr{L}_1$ is countably paracompact. Consider any element of $\phi\,(IR\,(\sigma, \mathscr{L}_2))$, $\phi\,(\nu)$. Set $\phi\,(\nu) = \mu$. Note to show $\mu \in IR\,(\sigma, \mathscr{L}_1)$, since $\mu \in IR\,(\mathscr{L}_1)$, it suffices to show for every sequence in $\mathscr{L}_1$, $<A_n>$, if $<A_n>$ is decreasing and $\lim_n A_n = \phi$, then $\lim_n \mu <A_n> = 0$. Accordingly, consider any sequence in $\mathscr{L}_1$, $<A_n>$, such that... and show $\lim_n \mu <A_n> = 0$. Assume $\lim_n \mu\,(A_n) \neq 0$.

Since $<A_n>$ is in $\mathscr{L}'_1$ and $<A_n>$ is decreasing and $\lim_n A_n = \phi$ and $\mathscr{L}'_1$ is countably paracompact, there exists a sequence in $\mathscr{L}_1$, $<C_n>$, such that for every n, $A_n \subset C'_n$ and $<C'_n>$ is decreasing and $\lim_n C'_n = \phi$. Consider any such $<C_n>$.

Note for every n, since $A_n \subset C'_n$ and $\mathscr{L}_1$ is normal, there exist two elements of $\mathscr{L}_1$, D_n, E_n, such that $A_n \subset D'_n$ and $C_n \subset E'_n$ and $D'_n \cap E'_n = \phi$; consider any such D_n, E_n; then $A_n \subset D'_n \subset E_n \subset C'_n$.

Hence for every n, $\bigcap_{i=1}^{n} A_i \subset \bigcap_{i=1}^{n} D'_i \subset \bigcap_{i=1}^{n} E_i \subset \bigcap_{i=1}^{n} C'_i$; hence, since $<A_m>$ is decreasing and $<C'_m>$ is decreasing, $A_n \subset \bigcap_{i=1}^{n} D'_i \subset \bigcap_{i=1}^{n} E_i \subset C'_n$; note $\left[\bigcap_{i=1}^{n} D'_i = \left(\bigcup_{i=1}^{n} D_i \right)' \text{ and } \bigcup_{i=1}^{n} D_i \in \mathscr{L}_1 \right]$ and $\bigcap_{i=1}^{n} E_i \in \mathscr{L}_1$; set $\bigcup_{i=1}^{n} D_i = \hat{D}_n$ and $\bigcap_{i=1}^{n} E_i = \hat{E}_n$; then $A_n \subset \hat{D}'_n \subset \hat{E}_n \subset C'_n$; consequently $A_n \subset \overline{\hat{D}'_n} \subset \hat{E}_n \subset C'_n$.

Consider $<\overline{\hat{D}'_n}>$.

(α) Note $<\overline{\hat{D}'_n}>$ is in $\mathscr{L}_2$ and $<\overline{\hat{D}'_n}>$ is decreasing and, since for every n, $\overline{\hat{D}'_n} \subset C'_n$ and $\lim_n C'_n = \phi$, $\lim_n \overline{\hat{D}'_n} = \phi$. Hence, since $\nu \in IR\,(\sigma, \mathscr{L}_2)$, $\lim_n \nu\,(\overline{\hat{D}'_n}) = 0$.

(β) Since $\lim_n \mu\,(A_n) \neq 0$ and $\mu \in I\,(\mathscr{L}_1)$ and $<A_n>$ is decreasing, for every n, $\mu\,(A_n) = 1$; hence, since $A_n \subset \hat{D}'_n$, $\mu\,(\hat{D}'_n) = 1$; hence $\mu \in W\,(\hat{D}'_n)$; consequently $\nu\,(\overline{\hat{D}'_n}) = 1$. (See the observation preceding the definition of ϕ). Hence $\lim_n \nu\,(\overline{\hat{D}'_n}) = 1$.

(γ) Thus a contradiction has been reached. Consequently $\lim_n \mu\,(A_n) = 0$. Consequently $\phi\,(\nu) \in IR\,(\sigma, \mathscr{L}_1)$.

Hence $\phi\,(IR\,(\sigma, \mathscr{L}_2)) \subset IR\,(\sigma, \mathscr{L}_1)$.

(ii) Assume $t\,\mathscr{L}_1$ is $\mathscr{L}_1$-weakly countably bounded.

Consider any element of $IR(\sigma, \mathscr{L}_1)$, μ. Then, since ϕ is onto, by Proposition 4.1, there exists an element of $IR(\mathscr{L}_2)$, ν, such that $\phi(\nu) = \mu$. Consider any such ν. Show $\nu \in IR(\sigma, \mathscr{L}_2)$. Note to show $\nu \in IR(\sigma, \mathscr{L}_2)$, since $\nu \in IR(\mathscr{L}_2)$, it suffices to show for every sequence in $\mathscr{L}_2$, $<B_n>$, if $<B_n>$ is decreasing and $\lim_n B_n = \phi$, then $\lim_n \nu(B_n) = 0$. Accordingly, consider any sequence in $\mathscr{L}_2$, $<B_n>$, such that .. and show $\lim_n \nu(B_n) = 0$. Assume $\lim_n \nu(B_n) \neq 0$.

Since $t\mathscr{L}_1$ is $\mathscr{L}_1$-weakly countably bounded and $<B_n>$ is in $\mathscr{L}_2$ and $<B_n>$ is decreasing and $\lim_n B_n = \phi$, by the relevant definition, there exists a sequence in $\mathscr{L}_1$, $<A_n>$, such that for every n, $B_n \subset A_n$ and $<A_n>$ is decreasing and $\lim_n A_n = \phi$. Consider any such $<A_n>$. Then, since $\mu \in IR(\sigma, \mathscr{L}_1)$, $\lim_n \mu(A_n) = 0$. Consequently there exists a value of n, m, such that $\mu(A_m) = 0$. Consider any such m.

Now, note since $\lim_n \nu(B_n) \neq 0$ and $\nu \in I(\mathscr{L}_2)$ and $<B_n>$ is decreasing, $\nu(B_m) = 1$. Hence, by the definition of ϕ, $\mu \in \bar{B}_m^{IR(\mathscr{L}_1)}$. Further, note, since $\mu(A_m) = 0$, $\mu(A_m') = 1$. Consequently $\mu \in W(A_m)'$. Thus $W(A_m)'$ is a neighbourhood of μ. Consequently $B_m \cap W(A_m)' \neq \phi$. Finally, note, since $B_m \subset A_m \subset W(A_m)$, $B_m \cap W(A_m)' = \phi$. Thus a contradiction has been reached. Consequently $\lim_n \nu(B_n) = 0$. Consequently $\nu \in IR(\sigma, \mathscr{L}_2)$.

Thus $IR(\sigma, \mathscr{L}_1) \subset \phi(IR(\sigma, \mathscr{L}_2))$.

Application—Consider any topological space X such that X is $T_{3\frac{1}{2}}$ and weakly countably bounded and let $\mathscr{L}_1 = \mathscr{Z}$. Note $\mathscr{L}_1$ is countably paracompact. Then, by Theorem 4. 4, $\phi(IR(\sigma, \mathscr{L}_2)) = IR(\sigma, \mathscr{L}_1)$, that is $\phi(IR(\sigma, \mathscr{F}_R)) = IR(\sigma, \mathscr{Z})$; otherwise stated : $\phi(IR(\sigma, \mathscr{F}_R)) = \upsilon X$ (Gillman and Jerison[9]).

Next, we introduce a suitable setting for further applications of Theorem 4.4.

Consider the background of the definition of ϕ with $\mathscr{L}_2 = \mathscr{L}(\mathscr{R}(t\mathscr{L}_1))$.

Observation 1 : Consider any element of $IR(\mathscr{L}_1)$, μ. Since ϕ is onto (Proposition 4.1, there exists an element of $IR(\mathscr{L}_2)$, ν, such that $\phi(\nu) = \mu$. Consider any such ν and of the type considered in Proposition 4.1. Note

$S(\nu) = \cap \{B \in \mathscr{L}_2 / \nu(B) = 1\}$, by the definition of support,

$\subset \cap \{X \cap \overline{O_\lambda}^{IR(\mathscr{L}_1)} ; \lambda \in \Lambda\}$, (see the proof of Proposition 4.1),

$= X \cap \cap \{\overline{O_\lambda}^{IR(\mathscr{L}_1)} ; \lambda \in \Lambda\}$

$= X \cap \{\mu\}$.

Case 1—$\mu \in X$. Then there exists an element of X, x, such that $\mu = \mu_x$ (and, since $\mathscr{L}_1$ is separating, x is unique). Now, consider the element of $I(\mathscr{L}_2)$ which is concentrated at x and denote it by ν_x. Note $\nu = \nu_x$. (See the proof of Corollary 3.4). Hence $\{x\} \subset S(\nu)$. Moreover, since $S(\nu) \subset X \cap \{\mu\}$ and $\mu = \mu_x$ and $S(\mu) = \{x\}$, $S(\nu) \subset \{x\} = S(\mu)$. Thus $S(\nu) = \{x\} = S(\mu)$.

Case 2—$\mu \notin X$. Then $S(\mu) = \phi$. Moreover, since $S(\nu) \subset X \cap \{\mu\}$ and $X \cap \{\mu\} = \phi$, $S(\nu) = \phi$. Thus $S(\nu) = \phi = S(\mu)$.

Consequently $S(\nu) = S(\mu)$.

Observation 2 : Consider any element of $IR(\mathscr{L}_2)$, ν. Further, consider $\phi(\nu)$ and set $\phi(\nu) = \mu$.

(α) Assume $S(\nu) = \phi$. Check $S(\mu) = \phi$. Assume $S(\mu) \neq \phi$. Consider any element of $S(\mu)$, x. Then, since $\mu \in IR(\mathscr{L}_1)$, $\mu = \mu_x$. Note $\nu = \nu_x$. (see Observation 1). Hence $S(\nu) \neq \phi$. Since this statement is false, the assumption is wrong. Consequently $S(\mu) = \phi$.

(β) Assume $S(\mu) = \phi$. Check $S(\nu) \neq \phi$. Assume $S(\nu) \neq \phi$. Consider any element of $S(\nu)$, x. Then, since $\nu \in IR(\mathscr{L}_2)$, $\nu = \nu_x$. Hence $\mu = \mu_x$ (See Proposition 2.7 (a)). Hence $S(\mu) \neq \phi$. Since this statement is false, the assumption is wrong. Consequently $S(\nu) = \phi$.

(γ) Thus $S(\nu) = \phi$ iff $S(\mu) = \phi$.

Corollary 4.5—(i) If $\mathscr{L}_1$ is countably paracompact, then $\mathscr{L}_1$ is replete $\Rightarrow$ $\mathscr{L}_2$ is replete.

(ii) If $t\,\mathscr{L}_1$ is $\mathscr{L}_1$-weakly countably bounded, then $\mathscr{L}_2$ is replete $\Rightarrow$ $\mathscr{L}_1$ is replete.

PROOF : (i) Assume $\mathscr{L}_1$ is countably paracompact. Further, assume $\mathscr{L}_1$ is replete. Note to show $\mathscr{L}_2$ is replete, according to the definition of repleteness, it suffices to show for every element of $IR(\mathscr{L}_2)$, ν, if $\nu \in IR(\sigma, \mathscr{L}_2)$, then $S(\nu) \neq \phi$. Accordingly, assume $IR(\sigma, \mathscr{L}_2) \neq \phi$ and consider any element of $IR(\sigma, \mathscr{L}_2)$, ν. Since $\mathscr{L}_1$ is countably paracompact, by [Theorem 4.4, (i)], $\phi(IR(\sigma, \mathscr{L}_2)) \subset IR(\sigma, \mathscr{L}_1)$. Consequently $\phi(\nu) \in IR(\sigma, \mathscr{L}_1)$. Hence, since $\mathscr{L}_1$ is replete, $S(\phi(\nu)) \neq \phi$. Then, by Observation 2, $S(\nu) \neq \phi$. Consequently $\mathscr{L}_2$ is replete.

(ii) Assume $t\,\mathscr{L}_1$ is $\mathscr{L}_1$-weakly countably bounded. Further, assume $\mathscr{L}_2$ is replete. Note to show $\mathscr{L}_1$ is replete, according to the definition of repleteness, it suffices to show ... Since $\mathscr{L}_1$ disjunctive, $IR(\sigma, \mathscr{L}_1) \neq \phi$. Consider any element of $IR(\sigma, \mathscr{L}_1)$, μ. Since $t\mathscr{L}_1$ is $\mathscr{L}_1$-weakly countably bounded, by (Theorem 4.4, (ii)), $IR(\sigma, \mathscr{L}_1) \subset \phi(IR(\sigma, \mathscr{L}_2))$. Consequently $\mu \in \phi(IR(\sigma, \mathscr{L}_2))$. Hence there exists an element of $IR(\sigma, \mathscr{L}_2)$, ν, such that $\phi(\nu) = \mu$. Consider any such ν. Then, since $\mathscr{L}_2$ is replete, $S(\nu) \neq \phi$. Moreover, by Observation 1, $S(\nu) = S(\mu)$. Consequently $S(\mu) \neq \phi$. Consequently $\mathscr{L}_1$ is replete.

Application—Consider any topological space X such that X is $T_{3\frac{1}{2}}$ and let $\mathscr{L}_1 = \mathscr{Z}$ and $\mathscr{L}_2 = \mathscr{F}_R$.

(a) Then, by (Corollary 4.5, (i)), $\mathscr{Z}$ is replete $\Rightarrow$ $\mathscr{F}_R$ is replete; otherwise stated: X is realcompact $\Rightarrow$ $\mathscr{F}_R$ is replete.

(b) If X is weakly countably bounded, then, by (Corollary 4.5, (ii)), $\mathscr{F}_R$ is replete $\Rightarrow$ X is realcompact.

(c) Consequently, if X is weakly countably bounded, then X is realcompact $\Leftrightarrow$ $\mathscr{F}_R$ is replete.

(d) X is Lindelöf $\Rightarrow$ $\mathscr{Z}$ is Lindelöf $\Rightarrow$ $\mathscr{Z}$ is replete $\Rightarrow$ $\mathscr{F}_R$ is replete. (See (a)).

§5. *Definition*—Consider any topological space X such that X is $T_{3\frac{1}{2}}$. X is c-realcompact iff for every element of βX, μ, if $\mu \in \beta X - X$, then there exists a sequence in $\mathscr{K}(\beta X)$, $<K_n>$, such that $<K_n>$ is decreasing and $\mu \in \bigcap_n K_n \subset \beta X - X$.

Remark : This definition is given by Hardy and Woods[10] and is proved by them to be equivalent to the definition given by Dykes[6].

The following theorem describes the relationship that repleteness of $\mathscr{F}_R$ (for a $T_{3\frac{1}{2}}$ space X) bears to c-realcompactness of X.

Theorem 5.1—Consider any topological space X such that X is $T_{3\frac{1}{2}}$. If $\mathscr{F}_R$ is replete, then for every element of βX, μ, if $\mu \in \beta X - X$, then there exists a sequence in $\mathscr{L}(\mathscr{K}(\beta X))$, $<K_n>$, such that $<K_n>$ is decreasing and $\mu \in \bigcap_n K_n \subset \beta X - X$.

Proof : Assume $\mathscr{F}_R$ is replete. Further, assume $\beta X - X \neq \phi$ and consider any element of $\beta X - X, \mu$. Since $\mu \in \beta X$ $(= IR(\mathscr{Z}))$ and ϕ is onto (by Proposition 4.1), there exists an element of $IR(\mathscr{F}_R)$, ν, such that $\phi(\nu) = \mu$. Consider any such ν. Then, since $\mu \in \beta X - X$, by Proposition 2.7, $\nu \in IR(\mathscr{F}_R) - X$. Hence, since $\mathscr{F}_R$ is replete, $\nu \notin IR(\sigma, \mathscr{F}_R)$. Hence there exists a sequence in $\mathscr{F}_R$, $<F_n>$, such that $<F_n>$ is decreasing and $\lim_n F_n = \phi$ and $\lim_n \nu(F_n) \neq 0$. Consider any such $<F_n>$. Since $<F_n>$ is in $\mathscr{F}_R$, on the basis of (Proposition 2.4, (ii)), there exists a sequence in $\mathscr{L}(\mathscr{K}(\beta X))$, $<K_n>$, such that for every n, $F_n = X \cap K_n$. Consider any such $<K_n>$. Since $<F_n>$ is decreasing, assume $<K_n>$ is also decreasing, without loss of generality. Now, show $\bigcap_n K_n \subset \beta X - X$. Note $(\bigcap_n K_n) \cap X = \bigcap (X \cap K_n) = \bigcap_n F_n = \phi$, since $<F_n>$ is decreasing and $\lim_n F_n = \phi$. Hence $\bigcap_n K_n \subset \beta X - X$. Finally, show $\mu \in \bigcap_n K_n$. Assume $\mu \notin \bigcap_n K_n$. Then there exists a value of n, m, such that $\mu \notin K_m$. Consider any such m. Then, since $\lim_n \nu(F_n) \neq 0$ and $\nu \in I(\mathscr{F}_R)$ and $<F_n>$ is decreasing, $\nu(F_m) = 1$. Hence, by the definition of ϕ, $\mu = \phi(\nu) \in \overline{F_m}^{\beta X}$.

Since $\mu \notin K_m$ and K_m is closed in βX, K'_m is a neighbourhood of μ. Consequently $F_m \cap K'_m \neq \phi$. Further, note, since $F_m = X \cap K_m$, $F_m \cap K'_m = \phi$. Thus a contradiction has been reached. Consequently $\mu \in \bigcap_n K_n$. Summarizing: $<K_n>$ is in $\mathscr{L}(\mathscr{Z}(\beta X))$ and $<K_n>$ is decreasing and $\mu \in \bigcap_n K_n \subset \beta X - X$.

The preceding theorem suggests the following :

Definition— X is strongly c-realcompact iff $\mathscr{F}_R$ is replete.

Corollary 5.2—If X is realcompact, then X is strongly c-realcompact.

PROOF : Assume X is realcompact. Then $\mathscr{F}_R$ is replete. (See Application following Corollary 4.5.) Hence, by the preceding definition, X is strongly c-realcompact.

The following definitions will be needed in Corollary 6.3.

Definition 1—Consider any set X and any lattice of subsets of X, $\mathscr{L}$. $\mathscr{L}$ is almost replete iff for every element of $I(\mathscr{L})$, μ, if $\mu \in IR(\mathscr{L}') \cap I(\sigma^*, \mathscr{L})$, then $S(\mu) \neq \phi$.

Definition 2—Consider any topological space X. X is almost realcompact iff $\mathscr{F}$ is almost replete[7].

Corollary 5.3—Consider any topological space X such that X is $T_{3\frac{1}{2}}$. If X is almost-realcompact, then X is strongly c-realcompact.

PROOF : Assume X is almost-realcompact. Note to show X is strongly c-realcompact, according to the relevant definition, it suffices to show $\mathscr{F}_R$ is replete, that is, for every element of $IR(\mathscr{F}_R)$, μ, if $\mu \in IR(\sigma, \mathscr{F}_R)$, then $S(\mu) \neq \phi$. Since $\mathscr{F}_R$ is disjunctive, $IR(\sigma, \mathscr{F}_R) \neq \phi$. Consider any element of $IR(\sigma, \mathscr{F}_R)$, μ, and show $S(\mu) \neq \phi$. Since $\mu \in IR(\mathscr{F}_R)$ and $\mathscr{F}_R \in \mathscr{F}$, there exists an element of $IR(\mathscr{F})$, ν, such that $\nu/\mathscr{A}(\mathscr{F}_R) = \mu$. Consider any such ν. Then $S(\nu) \subset S(\mu)$. Hence to show $S(\mu) \neq \phi$, it suffices to show $S(\nu) \neq \phi$. Since $\nu \in I(\mathscr{F})$, $\nu \in I(\mathscr{O})$. Hence there exists an element of $IR(\mathscr{O})$, λ, such that $\nu \leqq \lambda$ on $\mathscr{O}$. Consider any such λ. Then $\lambda \leqq \nu$ on $\mathscr{F}$. Since X is $T_{3\frac{1}{2}}$, $\mathscr{F}$ is regular. Consequently $S(\nu) = S(\lambda)$. Hence to show $S(\nu) \neq \phi$, it suffices to show $S(\lambda) \neq \phi$. Note to show $S(\lambda) \neq \phi$, since $\lambda \in IR(\mathscr{O})$ and X is almost-realcompact, it suffices to show $\lambda \in I(\sigma^*, \mathscr{F})$. Accordingly, consider any sequence in $\mathscr{F}$, $<F_n>$. such that $<F_n>$ is decreasing and $\lim_n F_n = \phi$ and show $\lim_n \lambda(F_n) = 0$. Note for every n,

$$\lambda(F_n) = \lambda(F_n^0), \text{ since } \lambda \in IR(\mathscr{O})$$

$$\leqq \lambda(\overline{F_n^0})$$

(*equation continued on p.* 1064)

$$\leqq \nu(\overline{F_n^0}), \text{ since } \lambda \leqq \nu \text{ on } \mathscr{F}$$

$$= \mu(\overline{F_n^0}), \text{ since } \nu/\mathscr{K}(\mathscr{F}_R) = \mu.$$

Now, consider $<\overline{F_n^0}>$. Note $<\overline{F_n^0}>$ is in $\mathscr{F}_R$ and, since $<F_n>$ is decreasing, $<\overline{F_n^0}>$ is decreasing, and, since for every n, $\overline{F_n^0} \subset F_n$ and $\lim_n F_n = \phi$, $\lim_n \overline{F_n^0} = \phi$. Hence, since $\mu \in I(\sigma^*, \mathscr{F}_R)$, $\lim_n \mu(\overline{F_n^0}) = 0$. Consequently $\lim_n \lambda(F_n) = 0$.

Consequently X is strongly c-realcompact.

The following theorem describes the relationship that c-realcompactness of X bears to repleteness of $\mathscr{F}_R$.

Lemma 5.4 (Corollary 2.2 of Bachman and Stratigos[3])—Consider any set X and any lattice of subsets of X, $\mathscr{L}$, such that $\mathscr{L}$ is separating and disjunctive. Then $\mathscr{L}$ is replete iff for every element of $IR(\mathscr{L})$, μ, if $\mu \in IR(\mathscr{L}) - X$, then there exists a sequence in $W(\mathscr{L})$, $<W(L_n)>$, such that $<W(L_n)>$ is decreasing and $\mu \in \bigcap_n W(L_n) \subset IR(L) - X$.

Theorem 5.5—Consider any topological space X such that X is $T_{3\frac{1}{2}}$. If X is c-realcompact and weakly countably bounded, then F_R is replete.

PROOF: Assume X is c-realcompact and weakly countably bounded. Note to show $\mathscr{F}_R$ is replete, since $\mathscr{Z}$ is replete $\Rightarrow \mathscr{F}_R$ is replete (Application following Corollary 4.5), it suffices to show $\mathscr{Z}$ is replete. Further, note to show $\mathscr{Z}$ is replete, according to Lemma 5.4, it suffices to show for every element of βX ($= IR(\mathscr{Z})$), μ, if $\mu \in \beta X - X$, then there exists a sequence in $W(\mathscr{Z})$, $<W(Z_n)>$, such that $<W(Z_n)>$ is decreasing and $\mu \in \bigcap_n W(Z_n) \subset \beta X - X$. Accordingly, assume $\beta X - X \neq \phi$ and consider any element of $\beta X - X$, μ.

Then, since X is c-realcompact, by the relevant definition, there exists a sequence in $\mathscr{R}(\beta X)$, $<K_n>$, such that $<K_n>$ is decreasing and $\mu \in \bigcap_n K_n \subset \beta X - X$. Consider any such $<K_n>$. Note for every n, since $K_n \in \mathscr{R}(\beta X)$, $K_n = \overline{K_n^0}$, and $\overline{K_n^0} \subset \overline{X \cap K_n^0}$, since X is dense in βX and K_n^0 is open in βX, $\subset \overline{X \cap K_n}$; since $K_n \in \mathscr{R}(\beta X)$, $X \cap K_n \in \mathscr{R}(X)$; set $X \cap K_n = E_n$.

Consider $<E_n>$. Note $<E_n>$ is in $\mathscr{R}(X)$ and, since $<K_n>$ is decreasing, $<E_n>$ is decreasing and, since $\bigcap_n K_n \subset \beta X - X$, $\lim_n E_n = \phi$. Hence, since X is weakly

countably bounded, there exists a sequence in $\mathscr{Z}$, $<Z_n>$, such that for every n, $E_n \subset Z_n$ and $<Z_n>$ is decreasing and $\lim_n Z_n = \phi$. Consider any such $<Z_n>$.

Then for every n, $K_n = \overline{K_n^0} \subset \bar{E}_n \subset \bar{Z} \subset W(Z_n)$. Consider $<W(Z_n)>$. Note $<W(Z_n)>$ is in $W(\mathscr{Z})$ and, since $<Z_n>$ is decreasing, $<W(Z_n)>$ is decreasing, and $\mu \in \bigcap_n K_n \subset \bigcap_n W(Z_n) \subset \beta X - X$, since $\lim_n Z_n = \phi$. Consequently $\mathscr{Z}$ is replete. (Thus a measure-theoretic proof of Dykes'[6] Theorem 3.1 has been obtained).

Consequently $\mathscr{F}_R$ is replete.

Note : The following statement is true : If X is extremally disconnected, then X is weakly countably bounded (Maese[12], Theorem 11).

Consequently the following statement is true : If X is c-realcompact and extremally disconnected, then $\mathscr{F}_R$ is replete.

We shall give a direct, measure-theoretic proof of this fact.

Assume X is c-realcompact and extremally disconnected. Note to show $\mathscr{F}_R$ is replete, according to Lemma 5.4, it suffices to show for every element of $IR(\mathscr{F}_R)$, ν, if $\nu \in IR(\mathscr{F}_R) - X$, then there exists a sequence in $W(\mathscr{F}_R)$, $<W(F_n)>$, such that $<W(F_n)>$ is decreasing and $\nu \in \bigcap_n W(F_n) \subset IR(\mathscr{F}_R) - X$. Accordingly, assume $IR(\mathscr{F}_R) - X \neq \phi$ and consider any element of $IR(\mathscr{F}_R) - X$, ν, and show there exists a sequence in $W(\mathscr{F}_R)$, etc. Consider $\phi(\nu)$ and set $\phi(\nu) = \mu$. Show $\mu \in \beta X - X$. Assume $\mu \notin \beta X - X$.

Then $\mu \in X$. Hence there exists an element of X, x, such that $\mu = \mu_x$, (and, since $\mathscr{Z}$ is separating, x is unique). Consider the element of $I(\mathscr{F}_R)$ which is concentrated at x and denote it by ν_x. Then $\nu = \nu_x$ (Review the proof of Corollary 3.4). Since $\nu \in IR(\mathscr{F}_R) - X$, this statement is false. Consequently $\mu \in \beta X - X$. Hence, since X is c-realcompact, there exists a sequence in $\mathscr{R}(\beta X)$, $<K_n>$, such that $<K_n>$ is decreasing and $\mu \in \bigcap_n K_n \subset \beta X - X$. Consider any such $<K_n>$.

Now, consider $<X \cap K_n>$. Since $<K_n>$ is in $\mathscr{R}(\beta X)$, on the basis of (Proposition 2.4, (i)), $<X \cap K_n>$ is in $\mathscr{R}(X)$, which is contained in $\mathscr{F}_R$. Consider $<W(X \cap K_n)>$. Since $<K_n>$ is decreasing, $<W(X \cap K_n)>$ is decreasing. Moreover, since $\bigcap_n K_n \subset \beta X - X$, $\bigcap_n W(X \cap K_n) \subset IR(\mathscr{F}_R) - X$. Finally, show $\nu \in \bigcap_n W(X \cap K_n)$. Assume $\nu \notin \bigcap_n W(X \cap K_n)$. Then there exists a value of n, m, such that $\nu \notin W(X \cap K_m)$. Consider any such m. Then there exists an element of $\mathscr{F}_R$, B, such that $B \cap (X \cap K_m) = \phi$ and $\nu(B) = 1$. Consider any such B. Then, by the definition of ϕ, $\mu \in \bar{B}^{\beta X}$. Further, note, since $B \cap (X \cap K_m) = \phi$ and $B \subset X$, $B \cap K_m = \phi$. Hence $\bar{B}^{\beta X} \subset \overline{\beta X - K_m}^{\beta X} = \beta X - K_m^{0\beta X}$. Hence,

since $\mu \in \bar{B}^{\beta X}$, $\mu \notin K_m^{0\,\beta X}$.

Since X is extremally disconnected, by (Wallman[16], p. 284, Proposition 10.47), βX is extremally disconnected. Hence, since $K_m \in \mathscr{R}(\beta X)$, K_m is open (in βX). Hence $K_m^{0\,\beta X} = K_m$. Hence, since $\mu \notin K_m^{0\,\beta X}$, $\mu \notin K_m$. Since $\mu \in \bigcap_n K_n$, $\mu \in K_m$. Thus a contradiction has been reached. Consequently $\nu \in \bigcap_n W(X \cap K_n)$. Summarizing : $\langle W(X \cap K_n)\rangle$ is in $W(\mathscr{F}_R)$ and $\langle W(X \cap K_n)\rangle$ is decreasing and $\nu \in \bigcap_n W(X \cap K_n) \subset IR(\mathscr{F}_R) - X$.

Consequently $\mathscr{F}_R$ is replete.

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A NOTE ON SWAN MODULES

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1. INTRODUCTION

Swan modules determine a canonical subgroup of the locally free classgroup of a group ring. In fact, this subgroup is contained in the kernel group, i.e. Swan modules are free over maximal orders. In this note we consider Swan modules over certain non-maximal orders. Motivation for studying such Swan modules is provided by Galois module structure problem in Taylor[10] where a certain generalised Swan module is shown to be the obstruction to the freeness of ring of integers over its associated order.

Let G be a group of order n. We set $\Sigma = \sum_G = \sum_{g \in G} g$, the sum of all group elements in the group ring $\mathbf{Z}G$. For each integer s relatively prime to n, we define a Swan module

$$<s, \Sigma> = s \cdot \mathbf{Z}G + \Sigma \cdot \mathbf{Z}G$$

a two-sided locally free $\mathbf{Z}G$-ideal. Swan modules were introduced in Swan[7]. We remark that some authors use a different definition of Swan module, for instance see Gruenburg and Linnell[4].

Each Swan module determines a class of $Cl(\mathbf{Z}G)$, the locally free classgroup of $\mathbf{Z}G$. It is well known that the set of all Swan classes form a finite subgroup $T(\mathbf{Z}G)$ of $Cl(\mathbf{Z}G)$, called the Swan subgroup of $Cl(\mathbf{Z}G)$ (cf. Curtis and Reiner[2].) The locally free classgroup $Cl(\mathbf{Z}G)$ is defined as the subgroup of elements of zero rank in $K_0(\mathbf{Z}G)$, the Grothendieck group of finitely generated, locally free $\mathbf{Z}G$-modules.

We now present the Galois module structure problem discussed in Taylor[10]. For a number field M, we write $\mathcal{O}_M$ for its ring of integers and for any ring R we write $(a, b) R$ for the right ideal $aR + bR$. Let K be a quadratic imaginary number field with discriminant less than -4. Moreover, assume that prime 2 splits in $K/\mathbf{Q}$. Let $\mathcal{P} = \lambda\mathcal{O}_K$ denote a non-ramified, principal prime ideal of $\mathcal{O}_K$, where $\lambda \equiv \pm 1 \bmod 4\mathcal{O}_K$. We fix positive integers $r > m$ and let N (respectively, L) denote K ray classifield mod $4\mathcal{P}^{m+r}$ (respectively, $4\mathcal{P}^r$). Let $\Gamma = Gal(N/L)$ and $\mathcal{R} = \{x \in L\Gamma : \mathcal{O}_N \cdot x \subseteq \mathcal{O}_N\}$, the associated order of the extension N/L in $L\Gamma$.

For $s \in Z$ with $(s, \lambda)\, \mathcal{O}_K = \mathcal{O}_K$, we define a locally free $\mathcal{A}$-ideal, $I_s = (s, \lambda^{-m} \sum_{\Gamma})\, \mathcal{A}$. We shall call I_s an elliptic Swan module (the use of 'elliptic' would become clear in Section 3). Taylor[10] showed that $\mathcal{O}_N$ is a free $\mathcal{A}$-module if, and only if the elliptic Swan module I_2 is $\mathcal{A}$-free.

In Srivastav[6] it is shown that $I_s = (s, \sum_{\Gamma})\, \mathcal{A}$ and therefore, it is obtained from the Swan module $(s, \sum_{\Gamma})\, Z\Gamma$ by an extension of rings. Thus, if $\mathcal{P}$ splits in K/Q then Γ is cyclic so that $T(Z\Gamma) = 0$ (cf. Swan[7]). In that case $(2, \sum_{\Gamma})\, Z\Gamma$ is $Z\Gamma$-free (since Γ is abelian, the Eichler condition is satisfied) and so I_2 is $\mathcal{A}$-free. Taylor[9] had shown that $\mathcal{O}_N$ is $\mathcal{A}$-free if $\mathcal{P}$ splits in K/Q without using Swan modules. So in the sequel we assume that $\mathcal{P}$ is inert in K/Q and set λ equal to an odd rational prime p. In this case Γ is a non-cyclic group of order p^{2m}.

Next, we note that the Z-order $Z\Gamma + Z\,(p^{-m} \sum_{\Gamma})$ is contained in $\mathcal{A}$. More generally, we let G be any abstract group of order p^k. For each integer j, $0 \leqslant j < k$ we consider the Z-order $\Lambda_j = ZG + Z\,(p^{-j} \Sigma)$. The Swan subgroup $T(ZG)$ maps, by an extension of rings, onto a subgroup $T(\Lambda_j)$, determined by Swan modules $(s, p^{-j} \Sigma)\, \Lambda_j$ with $p \nmid s$, in the locally free classgroup $Cl\,(\Lambda_j)$. From Taylor's theorem[8] that $T(ZG)$ is a cyclic group of order p^{k-1}, we are able to show that $T(\Lambda_j)$ is a cyclic group of order p^{k-1-j} (cf. Theorem 2).

As a corollary, set $G = \Gamma$ and $m = 1$ to show that I_2 is $\mathcal{A}$-free. This led the author to state.

Theorem 1—The elliptic Swan module I_2 is a principal ideal of the associated order $\mathcal{A}$.

We[6] used transcendental means to show that for $p \equiv \pm 1 \bmod 8$ the elliptic Swan module is, indeed, $\mathcal{A}$-free. The hypothesis $p \equiv \pm 1 \bmod 8$ was introduced in Srivastav[6] since in that case we could use the Lubin-Tate formal group law of the Fueter model in describing the local Galois module structure (cf. remark on page 173 of Cassou-Noguès and Taylor[1]). We shall employ the technical device of relative Lubin-Tate formal groups in removing this hypothesis in Section 3 to complete the proof of Theorem 1.

2. The Swan Subgroup

We keep the notation of Section 1. Let G be a group of order n. For each positive integer f that divides n, we define a Z-order

$$\Lambda_{(f)} = ZG + Z \cdot f^{-1} \Sigma \qquad \ldots(2.1)$$

in the group algebra QG. We should note that $\Lambda_{(f)}$ is Z-torsion free and finitely generated as a Z-module.

Let us fix f. For each integer s relatively prime to n, we define a $\Lambda(f)$-ideal

$$<s, f^{-1}\Sigma>(f) = (s, f^{-1}\Sigma)\,\Lambda(f). \qquad \text{...(2.2)}$$

We call this ideal a Swan module of $\Lambda(f)$ in the view of the following,

Lemma 1—With the above notation,

$$<s, f^{-1}\Sigma>(f) = (s, \Sigma)\,\Lambda(f).$$

PROOF : Clearly, $<s, f^{-1}\Sigma>(f) \supseteq (s, \Sigma)\,\Lambda(f)$. It suffices to show the equality locally at each prime q. If $q \mid s$, then $f \in \mathbf{Z}_q^x$ and we obtain the desired equality. On the other hand, if $q \nmid s$, then $s \in \mathbf{Z}_q^x$ and both ideals equal $\Lambda(f)_q$.

Thus $<s, f^{-1}\Sigma>(f)$ is a locally free $\Lambda(f)$-ideal obtained from the usual Swan module $<s, \Sigma>$ by extension of rings. It, therefore, determines a class, $<s, \Sigma>(f)$ in $Cl(\Lambda(f))$. We denote the set of all swan classes in $Cl(\Lambda(f))$ by $T(\Lambda(f))$. The inclusion

$$i : ZG \hookrightarrow \Lambda(f)$$

induces a surjective homomorphism

$$i_* : Cl(\mathbf{Z}G) \to Cl(\Lambda(f))$$

such that $T(\Lambda(f)) = i_*(T(\mathbf{Z}G))$. Hence $T(\Lambda(f))$ is a subgroup of $Cl(\Lambda(f))$ and we shall call $T(\Lambda(f))$, the Swan subgroup of $Cl(\Lambda(f))$.

For convenience we shall write $\Lambda = \Lambda(1) = ZG$. Let ϵ be the augmentation map of QG and we denote its restriction on $\Lambda(f)$ by ϵ_f. Now, we consider a fiber diagram

$$\begin{array}{ccc} \Lambda(f) & \xrightarrow{\epsilon_f} & \mathbf{Z} \\ \theta_f \big\downarrow & & \big\downarrow \phi_f \\ \dfrac{\Lambda(f)}{(f^{-1}\Sigma)} & \xrightarrow{\bar{\epsilon}_f} & \dfrac{\mathbf{Z}}{f^{-1}n\mathbf{Z}} \end{array} \qquad \text{...(2.3)}$$

where $\bar{\epsilon}_f$ is induced by ϵ_f and ϕ_f, θ_f are quotient maps.

The inclusion $i : \Lambda \hookrightarrow \Lambda(f)$ induces an isomorphism

$$i' : \frac{\Lambda}{(\Sigma)} \xrightarrow{\sim} \frac{\Lambda(f)}{(f^{-1}\Sigma)}. \qquad \text{...(2.4)}$$

The fact that $\Lambda/(\Sigma)$ is a finitely generated free $\mathbf{Z}$-module shows that $\Lambda(f)/(f^{-1}\Sigma)$ is also a $\mathbf{Z}$-order. For $f = 1$, the fiber diagram (2.3) was considered by Ullom[11]. This fiber diagram allows us to study the relation between the K-theory of $\Lambda(f)$ and that

of Z, $\Lambda_{(f)}/(f^{-1}\Sigma)$ and $Z/f^{-1}nZ$. In particular, there is an exact Mayer-Vietoris sequence of Reiner and Ullom[5].

$$K_1(Z) \oplus K_1\left(\frac{\Lambda_{(f)}}{(f^{-1}\Sigma)}\right) \to K_1\left(\frac{Z}{f^{-1}nZ}\right) \xrightarrow{\partial_f} D(\Lambda_{(f)}) \to 0 \qquad \ldots(2.5)$$

where $D(\Lambda_{(f)})$ is the kernel group of $\Lambda_{(f)}$. We recall that $D(\Lambda_{(f)})$ is a subgroup of $Cl(\Lambda_{(f)})$ and for any ring R, $K_1(R) = \dfrac{GL(R)}{GL'(R)}$, where the general linear group $GL(R) = \varinjlim GL_n(R)$ and $GL'(R) =$ the commutator subgroup of $GL(R)$. Moreover, in (2.5) $K_1(Z)$ (respectively, $K_1(\frac{Z}{f^{-1}nZ})$) may be identified with $Z^x = \{\pm 1\}$ (respectively, $(\frac{Z}{f^{-1}nZ})^x$) via the determinant map.

For $f = 1$, in (2.7) of Ullom[11] it is shown that $\partial_1(s \bmod nZ) = [s, \Sigma]$. In exactly the same manner we obtain the following:

Proposition 1—The connecting homomorphism ∂_f in (2.5) is given by

$$\partial_f(s \bmod f^{-1}nZ) = [s, \Sigma]_{(f)}.$$

The exact sequence (2.5) may now be rewritten as

$$Z^x \oplus K_1\left(\frac{\Lambda_{(f)}}{(f^{-1}\Sigma)}\right) \to \left(\frac{Z}{f^{-1}nZ}\right)^x \to T(\Lambda_{(f)}) \to 0. \qquad \ldots(2.6)$$

Next, we note the commutative diagram

$$\begin{array}{ccc} Z \oplus \dfrac{\Lambda}{(\Sigma)} & \xrightarrow{(\phi_1,\, -\bar{\epsilon}_1)} & \dfrac{Z}{nZ} \\ {\scriptstyle (id,\, i')}\downarrow & & \downarrow{\scriptstyle \pi_f} \\ Z \oplus \dfrac{\Lambda_{(f)}}{(f^{-1}\Sigma)} & \xrightarrow{(\phi_f,\, -\bar{\epsilon}_f)} & \dfrac{Z}{f^{-1}nZ} \end{array} \qquad \ldots(2.7)$$

where π_f is the quotient map.

From (2.7) using the functoriality of K_1 and the exactness of (2.6) together with Proposition 1 and Lemma 1 we obtain a commutative diagram with exact rows

$$\begin{array}{ccccccc} Z^x \oplus K_1\left(\dfrac{\Lambda}{(\Sigma)}\right) & \longrightarrow & \left(\dfrac{Z}{nZ}\right)^x & \xrightarrow{\partial_1} & T(\Lambda) & \to & 0 \\ {\scriptstyle (id,\, i'_*)}\downarrow & & {\scriptstyle \pi_f}\downarrow & & {\scriptstyle i_*}\downarrow & & \\ Z^x \oplus K_1\left(\dfrac{\Lambda_{(f)}}{(f^{-1}\Sigma)}\right) & \longrightarrow & \left(\dfrac{Z}{f^{-1}nZ}\right)^x & \xrightarrow{\partial_f} & T(\Lambda_{(f)}) & \to & 0. \end{array} \qquad \ldots(2.8)$$

In particular, we have

$$\mathrm{Ker}\,(\partial_f) = \pi_f\,(\mathrm{Ker}\,(\partial_1). \qquad \ldots(2.9)$$

Thus, if $T(\Lambda)$ is explicitly known then $T(\Lambda_{(f)})$ can also be calculated explicitly. For instance, if G is a cyclic group then we known that $T(\Lambda)$ is trivial so that $T(\Lambda_{(f)})$ is also trivial. As another example, we consider the case of p-groups. We remark that only in this section p may equal 2.

Let G be a non-cyclic p-group so that $n = p^k$. It is known that $T(\Lambda)$ is trivial for a dihedral 2-group and that $|T(\Lambda)| = 2$ for a generalized quaternion 2-group or a semidihedral 2-group (cf. Taylor[8]). Ullom conjectured that for a non-cyclic p-group which is not one of these types of 2-groups mentioned above

$$|T(\mathbf{Z}G)| = \begin{cases} p^{k-1}, & p \text{ odd} \\ 2^{k-2}, & p = 2. \end{cases}$$

Taylor used Fröhlich's hom description for $Cl(\Lambda)$, and introduced a modified version of the p-adic logarithm to give a remarkable proof of Ullom's conjecture [cf. (2.5) of Taylor[8]]. Taylor's theorem can now be generalised to describe $T(\Lambda_{(f)})$.

For convenience we now write Λ_j for $\Lambda_{(p^j)}$ where $0 \leqslant j \leqslant k$. Using (2.8) and so (2.9) we deduce from Taylor's theorem [(2.5) of Taylor[8]].

Theorem 2—Let G be a non-cyclic p-group of order p^k. If $p = 2$, assume that G is not a generalised quaternion, dihedral or semidihedral group. Let $0 \leqslant j < k$ (if $p = 2$, $0 \leqslant j < k - 1$).

(i) If $p \neq 2$, then $T(\Lambda_j)$ is a cyclic group of order p^{k-j-1} with $[1 + p, \Sigma]_{(p^j)}$ as a generator.

(ii) If $p = 2$, then $T(\Lambda_j)$ is a cyclic group of order 2^{k-j-2} with $[5, \Sigma]_{(2^j)}$ as a generator.

Now let us return to the Galois module structure problem of Section 1. We let p be an odd prime, inert in $K/\mathbf{Q}$ and set $G = \Gamma$, then $|G| = p^{2m}$. We also note that $\Lambda_m \subseteq \mathscr{H}$. In case $m = 1$, we conclude that $T(\Lambda_m) = 0$. Moreover, if $m = 2$ and p^2 is a Weiferich square, i.e. $2^{p-1} \equiv 1 \bmod p^2$, then $[2, \Sigma]_{(p^2)} = 0$. Thus we have,

Corollary 1—Let p be inert in $K/\mathbf{Q}$. If either $m = 1$ or $m = 2$ and p^2 is a Weiferich squre then I_2 is a principal $\mathscr{H}$-ideal.

An example of Wieferich square is 1093^2.

3. Elliptic Swan Modules

As in Taylor[10] and Srivastav[6] we consider the lattice $\Omega = \mathcal{O}_K$ in $\mathbf{C}$. We fix a primitive 4-division point ψ of $\mathbf{C}/\Omega$ such that 2ψ has annihilator 2Ω. We set a complex

number

$$t = \frac{12\ \mathscr{P}(2\psi)}{\mathscr{P}(\psi) - \mathscr{P}(2\psi)}$$

where $\mathscr{P}$ is the usual Weierstrass $\mathscr{P}$-function for Ω.

Let

$$\epsilon : y^2 = 4x^3 + tx^2 + 4x$$

be an elliptic curve with the identity of the group law at the origin $O = (0:0:1)$. We know[1] (Chapter XI) that $t^2 - 2^6$ is a unit in $K(4)$, the K ray classifield mod 4 $\mathcal{O}_K$. Moreover, the discriminant of $\mathscr{E}$ is $4(t^2 - 2^6)$. Thus $\mathscr{E}$ has good reduction at all odd primes. There is an isomorphism [cf. (4.10) Srivastav[6]] called the Fueter model

$$\xi : \mathbf{C}/\Omega \xrightarrow{\sim} \epsilon$$

given by

$$\xi(z) = \begin{cases} (T(z) : T_1(z) : 1), & z \neq 2\psi \\ (0 : 1 : 0), & z = 2\psi \end{cases}$$

where T and T_1 are two elliptic functions for Ω. We set

$$D(z) = \frac{T(z)}{T_1(z)}$$

an elliptic function for Ω.

Let $E = \dfrac{\mathcal{O}_K}{p^m\ \mathcal{O}_K}$, a finite ring. Then as in Taylor[10], the Galois group Γ and the group of p^m-division points of $\mathbf{C}/\Omega$ are both rank one free E-modules. We write both the E-actions exponentially as in Srivastav[6].

Let γ be an E-generator of Γ and α a primitive p^m-division point of $\mathbf{C}/\Omega$. In Srivastav[6] we defined the resolvent element ρ associated with α and γ by

$$\rho = p^{-m} \sum_{e \in E} \frac{D(\alpha^e + \psi)}{D(p^m\psi)} \gamma^{[e]}.$$

From (5.7) of Srivastav[6] we know that $p^m \rho \in \mathcal{O}_L\, \Gamma$.

Moreover, in (8.1) of Srivastav[6] we showed that if $p \equiv \pm 1 \bmod 8$, then $\rho \in \mathscr{H}$. We show

Theorem 3—The resolvent element ρ lies in the associated order $\mathscr{H}$.

From (3.6) and (5.17) of Srivastav[6] we obtain Theorem 1 as a consequence of

Theorem 3. In order to prove Theorem 3 we need to look at the formal group associated with the Fueter model in some detail.

We fix an embedding of $\bar{Q}$, a fixed algebraic closure of Q, in $\bar{Q}_p$, a fixed algebraic closure of Q_p, so that it corresponds to $\mathscr{P}$ for K. We write M' for the closure in $\bar{Q}_p$ of a field $M \subseteq \bar{Q}$. We note that $K(4)' = K'$ and $t \in K(4)$. We denote by P the maximal ideal of the ring of integers of $\bar{Q}_p$.

Let $\mathscr{E}'$ denote the elliptic curve $\mathscr{E}$ of the Fueter model (2.3) considered locally at P. This local elliptic curve $\mathscr{E}'$ admits complex multiplication and has good reduction modulo P. Let $\mathscr{E}_0'$ denote the kernel of reduction of $\mathscr{E}'$ modulo P. For convenience we write (x, y) for a point on $\mathscr{E}'$ with projective coordinates $(x : y : 1)$.

We know[1] (Chapter X) that there is a Lubin-Tate formal group law F' defined over $\mathcal{O}_{K'}$, for a uniformizer $p' \in \{\pm p\}$, where the parameter

$$t = \frac{2x}{y} \text{ on } F' \qquad \ldots(3.1)$$

is associated with the point (x, y) on $\mathscr{E}_0'$. Therefore, for a positive integer s and a primitive p^s-division point α_s of C/Ω, $(T(\alpha_s), T_1(\alpha_s)) \in \mathscr{E}_0'$ and the associated parameter $2D(\alpha_s)$ on F' is a primitive p^s-division point for F'.

Next, we note that $\dfrac{D(\alpha_s)}{D(\psi)} \in K(4\mathscr{P}^s)$ (cf. (5.6) of Srivastav[6]) and in addition, $\left[K'\left(\dfrac{D(\alpha_s)}{D(\psi)}\right) : K'\right] = [K(4\mathscr{P}^s)' : K']$. Thus we obtain

$$K(4\mathscr{P}^s)' = K'\left(\frac{D(\alpha_s)}{D(\psi)}\right). \qquad \ldots(3.2)$$

Now from (5.5) of Srivastav[6] and (6.8). Chapter IX of Cassou-Noguès and Taylor[1] we infer that $D(\psi) \in K(8)$ and $D^2(\psi) = (t+8)^{-1} \in K(4)$. This shows that $K'(D(\psi))/K'$ is an unramified extension of degree d, where $d/2$. Henceforth, we write K_n' for the unique unramified extension of K' of degree n so that

$$K_d' = K'(D(\psi)). \qquad \ldots(3.3)$$

In view of (3.2) we have

$$K_d'(D(\alpha_s)) = K(4\mathscr{P}^s)'(D(\psi)). \qquad \ldots(3.4)$$

From local classfield theory as in Taylor[10] there is a relative Lubin-Tate formal group law F'' on $\mathcal{O}_{K'}$, for a uniformizer p'' such that

$$K(4\mathscr{P}^s)' = K'(\omega_s) \qquad \ldots(3.5)$$

where ω_s is a primitive p^s-division point for F''. Combining (3.4) and (3.5) we obtain

$$K'_d(D(\alpha_s)) = K'_d(\omega_s). \qquad \ldots(3.6)$$

The finite ring E acts on p^m-division points of $\mathbf{C}/\Omega$ and also on p^m-division points of F''.

Proposition 2—Let α be a primitive p^m-division point of $\mathbf{C}/\Omega$. Then there exists

(i) a formal power series $\theta(X) \in \mathcal{O}_{K'_d}[[X]]$

and

(ii) a primitive p^m-division point ω of F'' such that

$$2D(\alpha^e) = \theta(\omega_{[e]}) \;\forall\, e \in E.$$

PROOF : We view both F' and F'' as relative Lubin-Tate formal group laws over $\mathcal{O}_{K'_d}$. From (1.2), Chapter I of de Shalit[3] we see that F' (respectively, F'') is a relative Lubin-Tate formal group law for $(p')^d$ (respectively, $(p'')^d$). For each positive integer s, let α_s (respectively, ω_s) be a primitive p^s-division point of $\mathbf{C}/\Omega$ (respectively, F'').

We know from (1.8), Chapter I of de Shalit[3] that $K'(D(\alpha_s))$ (respectively, $K'_d(\omega_s)$) is the classifield for K' to the subgroup $<(p')^d> \cdot (1 + \mathscr{P}^s)$ (respectively, $<(p'')^d> \cdot (1 + \mathscr{P}^s)$) of $(K')^*$. From (3.6) we deduce that

$$<(p)'^d> \cdot (1 + \mathscr{P}^s) = <(p'')^d> \cdot (1 + \mathscr{P}^s). \qquad \ldots(3.7)$$

Since (3.7) holds for each positive integer s we must have

$$(p')^d = (p'')^d. \qquad \ldots(3.8)$$

From (3.8) we conclude that F' and F'' are both relative Lubin-Tate formal group law for $(p')^d$. Hence by (1.5), Chapter I of de Shalit[3], F' and F'' are isomorphic formal group laws over $\mathcal{O}_{K'_d}$ and there exists a formal power series $\theta(X) \in \mathcal{O}_{K'_d}[[x]]$ such that

$$\theta((F''(X, Y)) = F'(\theta(X), \theta(Y)) \qquad \ldots(3.9)$$

and

$$\theta([a]_{F''}(X)) = [a]_{F'}(\theta(X)) \;\forall\, a \in \mathcal{O}_{K'}. \qquad \ldots(3.10)$$

In view of (3.10) there exists a primitive p^m-division point ω of F'' such that

$$2D(\alpha) = \theta(\omega). \qquad \ldots(3.11)$$

Moreover, applying (3.10) on (3.11) we obtain

$$2D(\alpha^e) = \theta(\omega^{[e]}) \;\forall\, e \in E$$

proving the proposition.

Remark : In the case that $d = 1$, from (3.8) we obtain that $p' = p''$ and then $\theta(X) = X$. Indeed, this is the case for $p \equiv \pm 1 \bmod 8$ as seen in (8.2) of Srivastav[6], which is

$$2D(\alpha^e) = \omega^{[e]} \;\forall\, e \in E. \qquad \ldots(3.12)$$

We also note that if $d = 2$ then $p' = -p''$.

Now we can prove Theorem 3.

Proof of Theorem 3—As in the proof of (8.1) of Srivastav[6] it suffices to show that $\rho \in \mathscr{H}_q$ whenever q is a prime of $\mathcal{O}_L$ such that $q \mid \mathscr{F}$. For such a prime q of $\mathcal{O}_L$ we fix the embedding of $\bar{\mathrm{Q}}$ in $\bar{\mathrm{Q}}_p$ so that it corresponds to q_N over N where q_N is the unique prime of $\mathcal{O}_N$ with $q = q_N \cap \mathcal{O}_L$.

We write $\mathscr{H}'$ for $\mathscr{H}_q$. From (3.3) and (3.6) of Srivastav[6] we note that

$$\mathscr{H}' = \mathcal{O}_{L'} \cdot 1_\Gamma + \sum_{i=0}^{p^{2m}-2} \mathcal{O}_{L'} \cdot \sigma_i \qquad \ldots(3.13)$$

where

$$\sigma_i = p^{-m} \cdot \sum_{e \in E} (\omega^{[e]})^i \, \gamma_{[e]} - 1_\Gamma) \in \mathscr{H}' \;\forall i \geqslant 0.$$

We set

$$L'' = L'(D(\psi)) \qquad \ldots(3.14)$$

and

$$\mathscr{H}'' = \mathcal{O}_{L''} \cdot 1_\Gamma + \sum_{i=0}^{p^{2m}-2} \mathcal{O}_{L''} \cdot \sigma_i \qquad \ldots(3.15)$$

an $\mathcal{O}_{L''}$-order in $L''\,\Gamma$. Since $\{1_\Gamma, \sigma_0, \sigma_1, \ldots, \sigma_{p^{2m}-2}\}$ forms an L'-basis of $L'\,\Gamma$ we deduce that

$$\mathscr{H}' = \mathscr{H}'' \cap L'\Gamma. \qquad \ldots(3.16)$$

Thus in order to prove Theorem 3 it suffices to show that $\rho \in \mathscr{H}''$ since $\rho \in L\Gamma$. We show that $\rho \in \mathscr{H}''$ by proceeding exactly as in the proof of (8.1) of Srivastav[6] and using Proposition 2 instead of (3.12).

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FIXED POINT THEOREMS FOR MULTIVALUED MAPPINGS

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Let X be a complete metric space and $B(X)$ the space of closed bounded subsets of X with the Hausdorff metric. Two general fixed point theorems for $T : X \to B(X)$ are proved. Then examples show that these theorems include known fixed point theorems and also yield new theorems.

Let (X, d) be a complete metric space. $B(X)$ denotes all closed bounded subsets of X with the Hausdorff metric ρ defined by

$$\rho(E, F) = \max [\sup_{x \in F} d(x, E), \sup_{x \in E} d(x, F)].$$

Also,

$$\rho(E, F) = \sup_{x \in X} | d(x, E) - d(x, F) | .$$

Theorem 1—Suppose $T : (X, d) \to (B(X), \rho)$ where X is complete and T is continuous. Then there exists p in X with $p \in Tp$ if and only if there exists a sequence $\{x_n\}$ in X with $x_{n+1} \in T x_n$ and $\sum_{n=1}^{\infty} d(x_n, x_{n+1}) < \infty$. In this case, $p = \lim_n x_n$.

Proof : If $p \in Tp$, let $x_n = p$ for every n. Suppose the condition holds. Then $\{x_n\}$ is a Cauchy sequence, so that $\lim x_n = p$ for some p in X. T is continuous implies $Tx_n \to Tp$. If $y \in Tp$, $d(p, y) \leqslant d(p, x_{n+1}) + d(x_{n+1}, y)$ and it follows that $d(p, Tp) \leqslant d(p, x_{n+1}) + d(x_{n+1}, Tp)$. Now $d(p, x_{n+1}) \to 0$ and

$$d(x_{n+1}, Tp) \leqslant \sup \{d(y, Tp) : y \in Tx_n\} \leqslant \rho(Tx_n, Tp) \to 0 \text{ as } n \to \infty.$$

Thus, $d(p, Tp) = 0$ and $p \in \overline{Tp} = Tp$.

Theorem 2—Suppose $T : (X, d) \to (B(x), \rho)$ where X is complete, T is continuous, and Tx is compact for each x in X. Assume that $\rho(Tx, Ty) \leqslant K(d(x, y))$ where $K : [0, \infty) \to [0, \infty)$, $K(0) = 0$, and K is non-decreasing. Then there exists p in X with $p \in Tp$ if and only if there exists x_0 in X with $\sum_{n=1}^{\infty} K^n (d(x_0, Tx_0)) < \infty$. In this case, we can choose $x_{n+1} \in Tx_n$ with $x_n \to p$.

(K is not assumed to be continuous and $K^2(t) = K(K(t))$.)

PROOF : If $p \in Tp$, $d(p, Tp) = 0$, $0 = K(0) = K^2(0)$, and $\sum_{n=1}^{\infty} K^n (d(p, Tp)) = 0$.

Suppose there exists x_0 such that $\sum_{n=1}^{\infty} K^n (d(x_0, Tx_0)) < \infty$. $y \to d(x_0, y)$ is continuous on the compact set Tx_0 implies there exists $x_1 \in Tx_0$ such that $d(x_1, x_0) = \min \{d(x_0, y) : y \in Tx_0\} = d(x_0, Tx_0)$. Similarly, Tx_1 is compact so there exists $x_2 \in Tx_1$ such that $d(x_2, x_1) = d(x_1, Tx_1)$. Thus, we obtain a sequence $\{x_n\}$ such that

$$x_{n+1} \in Tx_n \text{ and } d(x_{n+1}, x_n) = d(x_n, Tx_n).$$

Now, $d(x_n, Tx_{n-1}) = 0$ since $x_n \in Tx_{n-1}$. Since K is non-decseasing,

$$\begin{aligned} d(x_{n+1}, x_n) &= d(x_n, Tx_n) - 0 \\ &= d(x_n, Tx_n) - d(x_n, Tx_{n-1}) \\ &\leqslant \rho(Tx_n, Tx_{n-1}) \leqslant K(d(x_n, x_{n-1})) \\ &\leqslant K^2(d(x_{n-1}, x_{n-2})) \leqslant \ldots \\ &\leqslant K^n(d(x_1, x_0)) = K^n(d(x_0, Tx_0)). \end{aligned}$$

Applying Theorem 1, we obtain $p \in Tp$ where $p = \lim_n x_n$.

Remarks : K need only be defined on the range of d. If you replace your metric with an equivalent metric with $d(x, y) < 1$, then clearly Theorem 2 holds for K: $[0,1) \to [0, \infty)$. In many applications, X is a normed linear space and Tx is compact and convex for each x in X. To apply Theorem 2, one needs a non-decreasing function K and x in X with $\sum_{n=1}^{\infty} K^n (d(x, Tx)) < \infty$. The following examples satisfy these conditions and therefore illustrate the generality of Theorem 2 see Hicks[1]. For the details that are not obvious and not provided.

Example 1—Suppose $0 < \lambda < 1$. Let $K(t) = \lambda t$ for $t \geqslant 0$. Then $\rho(Tx, Ty) \leqslant K(d(x, y)) = \lambda d(x, y)$. $K^n(d(x, Tx)) = \lambda^n d(x, Tx)$ for any x in X. It is known that there exists p with $p \in Tp$ without assuming Tx is compact.

Example 2—Suppose T satisfies $\rho(Tx, Ty) \leqslant \phi(d(x, y))\, d(x, y)$ for all x, y in X, where $\phi : [0, \infty) \to [0, 1)$ and ϕ is non-decreasing. Then $K(t) = t\,\phi(t)$, K is non-decreasing, and K: $[0, \infty) \to [0, \infty)$. It follows by induction that

$$K^n(t) \leqslant t\,[\phi(t)]^n. \quad \text{Since } \phi(t) < 1, \ \sum_{n=1}^{\infty} K^n(t) < \infty.$$

Example 3—Consider $K(t) = t\,\phi(t)$ where $\phi : [0, \infty) \to [0, \infty)$, $\phi(t) \leqslant t$ for $t < 1$. If $t < 1$, it follows that $K^n(t) \leqslant t\,[\phi(t)]^n$. If K is non-decreasing Theorem 2 can be applied.

Example 4—$K(t) = t\,\phi(t)$ where ϕ: $[0, \infty) \to [0, \infty)$ and $\phi(\alpha t) \leqslant \alpha\,\phi(t)$ for $\alpha \in (0, 1]$. If $\phi(t) < 1$, $K^n(t) \leqslant (Kt)\,(\phi(t))^n$ for all $n \geqslant 2$.

Example 5—Assume K is non-decreasing, K is convex on $[0, 1)$ and $K(t) < t$ for all $0 < t < 1$. If $t < 1$, $K(t) < t$ so $K(t) = \alpha t$ for some $0 < \alpha < 1$. It can be shown that $K^n(t) \leqslant \alpha^n t$ for all n and thus $\sum_{n=1}^{\infty} K^n(t) < \infty$.

Theorem 3—Suppose $T: X \to B(X)$ where X is complete and Tx is compact for each x. Suppose $\rho(Tx, Ty) \leqslant [d(x, y)]^q$ where $q > 1$. If there exists x such that $t = d(x: Tx) < 1$, we can choose a sequence $\{x_n\}$ with $x_{n+1} \in Tx_n$ and $x_n \to p$ with $p \in Tp$.

PROOF : Let $K(t) = t^q$ for $t \geqslant 0$. $K(0) = 0$, K is increasing, $K(t) < t$ if $t < 1$, and K is convex. If $t = d(x, Tx) < 1$, $\sum_{n=1}^{\infty} K^n(t) < \infty$ from the previous example. T is continuous so Theorem 2 applies.

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AN ABSTRACT FIXED POINT THEOREM FOR MULTI-VALUED MAPPINGS

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In this paper we have proved a fixed point theorem for multi-valued non-expansive mappings in a metric space endowed with some convexity structure.

INTRODUCTION

Kirk[1] has proved a fixed point theorem for nonexpansive mappings over a closed convex subset of a reflexive Banach space satisfying normal structure. Penot[3] and Kirk[2], under different approaches, established an abstract version of the result of Kirk[1] in a bounded metric space. In this paper, modifying slightly the proofs of Kirk[2], we extend result for multi-valued mappings, generalizing also fixed point Theorem 3.1 of Samanta[4].

SOME NOTATIONS AND DEFINITIONS

In this paper we shall assume (X, d) to be a metric space, M be a bounded subset of X and $\mathscr{F}(X)$ be the collection of all nonempty subsets of X. If $x \in X$ and $r > 0$, then $B(x, r)$ denotes the closed ball with centre at x and radius r. Following Kirk[2], we put for any nonempty subset D of M:

$$\delta(D) = \text{Sup}\,\{d(u, v) : u, v \in D\}$$

$$r_u(D) = \text{Sup}\,\{d(u, v), v \in D\}, u \in D$$

$$r(D) = \text{Inf}\,\{r_u(D) : u \in D\}$$

$$h(D) = \begin{cases} r(D)/\delta(D), & \text{if } \delta(D) > 0 \\ 1, & \text{if } \delta(D) = 0. \end{cases}$$

Further for any two subsets A, B of M, the Hausdorff distance $H(A, B)$ between A and B is defined by

$$H(A, B) = \text{Inf}\,\{t > 0 \text{ such that for } a \in A,\ b \in B, \text{ there exists } a' \in A,\ b' \in B \text{ such that } d(a, b') < t \text{ and } d(a', b) < t\}.$$

Following Kirk[2], a class $\mathscr{S}$ of subsets of X is said to be normal if for each $D \in \mathscr{S}$, $\delta(D) > 0$ implies $h(D) \in (0, 1)$. The class $\mathscr{S}$ is said to be (countably) compact if

each (countable) subfamily of $\mathscr{S}$ which has finite intersection property has nonvoid intersection.

For the purpose of our theorem we shall take $\mathscr{S}$ to be a class of subsets of M which is countably compact, stable under intersections, normal and contains the closed balls of M.

Theorem—If $T : M \to \mathscr{F}(X)$ be a mapping such that

(a) $T(x) \cap M \neq \phi, \forall\ x \in M$

(b) $\forall\ x \in M,\ T(x) \cap \mathscr{S} = \{T(x) \cap D : D \in \mathscr{S}\}$

is a compact class of which each nonempty member is a compact subset of X,

(c) for any $G \in \mathscr{S}$ satisfying $T(\xi) \cap G \neq \phi,\ \forall\ \xi \in G$,

$$H(T(x) \cap G,\ T(y) \cap G) \leqslant d(x, y),\ \forall\ x, y \in G$$

then T has a fixed point in M.

PROOF : Modifying slightly the proof of Lemma of Kirk[2], we show that for each $\epsilon > 0$, there exists a nonempty set $M(\epsilon) \in \mathscr{S}$ such that $T(x) \cap M(\epsilon) \neq \phi, \forall\ x \in M(\epsilon)$ and for which $\delta(M(\epsilon)) \leqslant (h(M) + \epsilon)\,\delta(M)$. For this take $M(\epsilon) = M$ if $\delta(M) = 0$. Otherwise, construct $M(\epsilon)$ as follows : Let $\rho = (h(M) + \epsilon)\,\delta(M)$.

By the definition of h, the set $\mathscr{C} = \{z \in M : M \subset B(z, \rho)\}$ is nonempty. Let $\mathscr{F} = \{D \in \mathscr{S};\ \mathscr{C} \subset D,\ T(x) \cap D \neq \phi,\ \forall\ x \in D\}$. Order the family $\mathscr{F}$ by set-inclusion relation. Let $\tau = \{D_i\}_{i \in \Delta}$ be a decreasing chain in $\mathscr{F}$. Let $D_0 = \bigcap_{i \in L} D_i$. Then $D_0 \in \mathscr{S}$, and $C \subset D_0$. Further, since $\{D_i\}_{i \in \Delta}$ is decreasing, it follows that for each $x \in D_0$, the family $\{T(x) \cap D_i\}_{i \in \Delta}$ has finite intersection property. So, by hypothesis (b), $T(x) \cap (\bigcap_{i \in \Delta} D_i) \neq \phi$. i. e., $T(x) \cap D_0, \neq \phi$. Thus every decreasing chain in $\mathscr{F}$ has a lower bound. Therefore, by Zorn's Lemma, $\mathscr{F}$ has a minimal element L (say). Let $A = \mathscr{C} \cup T(L)$, where $T(L) = \bigcup_{x \in \Delta} (T(x) \cap L)$. Then $T(L) \subset L$. So, $\mathrm{Cov}(A) = \cap \{D \in \mathscr{S};\ A \subset D\} \subset L$. Also for $x \in \mathrm{Cov}(A)$, $T(x) \cap L \neq \phi$ and $T(x) \cap L \subset T(L)$. So, $T(x) \cap A \neq \phi$. Hence $T(x) \cap \mathrm{Cov}(A) \neq \phi$. Thus $\mathrm{Cov}(A) \in \mathscr{F}$. Since L is a minimal member of $\mathscr{F}$ and $\mathrm{Cov}(A) \subset L$, it follows that $\mathrm{Cov}(A) = L$. Let $M(\epsilon) = \{x \in L;\ L \subset B(x, \rho)\}$. Then $M(\epsilon) \neq \phi$ since $M(\epsilon) \supset C$. Let $x \in M(\epsilon)$. Then $T(x) \cap L \neq \phi$. Take $x' \in T(x) \cap L$. Let $O = L \cap B(x', \rho)$. Then $C \subset O$. Next, take $\eta \in O$. Then $\eta \in L$ and $d(x', \eta) \leqslant \rho$. Now, by (c),

$$\begin{aligned} H(T(x) \cap L,\ T(\eta) \cap L) &\leqslant d(x, \eta) \\ &\leqslant \rho\ (\because x \in M(\epsilon),\ \eta \in L). \end{aligned}$$

Since $T(x) \cap L$ and $T(\eta) \cap L$ are nonempty compact sets and $x' \in T(x) \cap L$, there exists $\eta' \in T(\eta) \cap L$ such that

$$d(x', \eta') \leqslant H(T(x) \cap L,\ T(\eta) \cap L) \leqslant \rho.$$

So $\eta' \in O$, Hence $T(\eta) \cap O \neq \phi$. Thus $O \in \mathscr{F}$ and $O \subset L$. Since L is a minimal element in $\mathscr{F}$, it follows that $O = L$. So $d(x', y) \leqslant \rho$. $\forall\ y \in L$. i. e., $B(x', \rho) \supset L$. So, $x' \in M(\epsilon)$, which implies $T(x) \cap M(\epsilon) \neq \phi$.

Now let $m = \{D \in \mathscr{S};\ D \neq \phi,\ T(x) \cap D \neq \phi,\ \forall\ x \in D\}$, and for each $D \in m$, Let $\delta_0\ (D) = \text{Inf}\ \{\delta(F);\ F \in m,\ F \subset D\}$. From now on, the proof runs similarly as in Kirk[2] and hence it is omitted.

Remark : The necessity of the condition (c) of the Theorem has been studied by Samanta[4].

Corollary 1 (Theorem 1 of Kirk[2])—Let (M, d) be a non-empty bounded metric space and suppose M contains a class $\mathscr{S}$ of subsets which is countably compact, stable under arbitrary intersections, and normal. Suppose further that $\mathscr{S}$ contains the closed balls of M. Then every nonexpansive mapping T of M into itself has a fixed point.

Corollary 2 (Theorem 3.1 of Samanta[4])—Let X be a reflexive Banach space and K, a bounded closed convex subset of X, possessing normal structure. If $\psi : K \to 2^X$ is a mapping such that

(a) $\psi(x) \cap K \neq \phi,\ \forall\ x \in K$

(b) for any closed convex subset G of K satisfying $\psi(\xi) \cap G \neq \phi,\ \forall\ \xi \in G$,

$$H(\psi(x) \cap G,\ \psi(y) \cap G) \leqslant \|x - y\|, \text{ whenever } x, y\ (\neq x) \in G.$$

Then ψ has a fixed point.

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ON LINEAR INDEPENDENCE OF SEQUENCES IN CONJUGATE BANACH SPACES

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The concepts of *minimal, *ω-linearly independent and * Λ-linearly independent sequences in conjugate Banach spaces have been introduced. The relationships among these concepts and the concepts of minimal, ω-linearly independent and Λ-linearly independent sequences have been established and counter examples to support the non-implications among them have been exihibited. Also weak*-bases have been characterized in terms of *ω-linearly independent sequuences and finally deficient *minimal sequences have been defined and discussed.

1. Introduction

The concepts of minimal sequences and ω-linearly independent sequences in Banach spaces were studied by many workers[1-7] and in more general spaces by Kamthan and Gupta[8,9] Λ-linearly independent sequences ($\Lambda = \{\lambda_i\}$ with $\lambda_i > 0$, $i \in \mathbb{N}$) were first considered by Erdös and Straus[2] and later by Singer[4].

The purpose of this paper is to study these concepts together with others like *minimal, *ω-linearly independent and *Λ-linearly independent sequences in conjugate Banach spaces. In fact, we consider here 28 implications and counter-implications among these concepts. In section 4, counter examples have been exhibited to support the non-implications among these concepts. The main results are established in Section 5, wherein among other results, it is shown that "Let $\{f_n\} \subset E^*$ be a *ω-linearly independent sequence. Then $\{f_n\}$ is a weak*-basis of E^* if and only if $\{f_n\}$ is a maximal *ω-linearly independent sequence". Also, a diagram containing 28 implications and counter-implications has been given. Finally, the concept of deficient *minimal sequences in conjugate Banach spaces has been defined in Section 6. It is proved that if E is separable and $\{f_n\} \subset E^*$ is a sequence which is not deficient *minimal, then for every sequence of positive integers $\{C_n\}$, there exisis a total sequence $\{g_n\} \subset E^*$ such that $\| f_n - g_n \| \leqslant C_n$, for each n.

2. Standard Definitions

Throughout E will denote a Banach space over the scalar field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$), E^* and E^{**}, respectively, the first and second conjugate spaces of E, π the canonical

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isomorphism of E into E^{**}, $[x_n]$ the closed linear span of $\{x_n\}$ in the norm topology and $\widetilde{[f_n]}$ the closed linear span of $\{f_n\}$ in the weak*-topology.

A sequence $\{x_n\} \subset E$ is complete in E if $[x_n] = E$ and $\{f_n\} \subset E^*$ is total on E if $f_n(x) = 0$ for all n implies $x = 0$. A system (x_n, f_n) $(\{x_n\} \subset E, \{f_n\} \subset E^*)$ is a biorthogonal system if $f_i(x_j) = \delta_{ij}$ for all i and j. A sequence $\{x_n\} \subset E$ is minimal if there exists a sequence $\{f_n\} \subset E^*$ such that (x_n, f_n) is a biorthogonal system. A sequence $\{x_n\} \subset E$ is ω-linearly independent if $\{\alpha_n\} \subset \mathbb{K}$, $\sum_{i=1}^{\infty} \alpha_i x_i = 0$ imply $\alpha_i = 0$ for all i. A sequence $\{x_n\} \subset E$ is Λ-linearly independent ($\Lambda = \{\lambda_i\}$ with $\lambda_i > 0$, $i \in \mathbb{N}$) if $|\alpha_i^{(n)}| \leqslant \lambda_i$, for each i and n, and $\lim_{n\to\infty} \sum_{i=1}^{\infty} \alpha_i^{(n)} x_i = 0$ imply $\lim_{n\to\infty} \alpha_i^{(n)} = 0$ for all i. A sequence $\{x_n\} \subset E$ is a basis in E if for every $x \in E$, there is a unique sequence $\{\alpha_n\} \subset \mathbb{K}$ such that $x = \sum_{i=1}^{\infty} \alpha_i x_i$, the convergence being in the norm topology of E. A sequence $\{f_n\} \subset E^*$ is a weak*-basis in E^* if for every $f \in E^*$, there is a unique sequence $\{\alpha_n\} \subset \mathbb{K}$ such that $f(x) = \sum_{i=1}^{\infty} \alpha_i f_i(x)$, for every $x \in E$.

It is well known that every minimal sequence is ω-linearly independent but not conversely (Singer[3], p. 50). Also, every minimal sequence is Λ-linearly independent ($\Lambda = \{\lambda_i\}$ with $\lambda_i > 0$, $i \in \mathbb{N}$) and every normalized Λ-linearly independent sequence with $\inf_i \lambda_i > 0$ is ω-linearly independent while the converse in each case is not true[2].

3. New Definitions

Definition 1—A sequence $\{f_n\} \subset E^*$ is said to be a *minimal sequence if there exists a sequence $\{x_n\} \subset E$, called an admissible sequence to the *minimal sequence, such that (x_n, f_n) is a biorthogonal system.

Definition 2—A sequence $\{f_n\} \subset E^*$ is said to be *ω-linearly indepenent if $\{\alpha_n\} \subset \mathbb{K}$, $\sum_{i=1}^{\infty} \alpha_i f_i(x) = 0$, for each $x \in E$ imply $\alpha_i = 0$, $i \in \mathbb{N}$.

Definition 3—Let $\Lambda = \{\lambda_i\}$ with $\lambda_i > 0$, $i \in \mathbb{N}$. A sequence $\{f_n\} \subset E^*$ is said to be *Λ-linearly independent if

$$\left.\begin{array}{l} |\alpha_i^{(n)}| \leqslant \lambda_i \ (i, n \in \mathbb{N}), \\ \lim_{n\to\infty} \sum_{i=1}^{\infty} \alpha_i^{(n)} f_i(x) = 0, \text{ for each } x \in E \end{array}\right\} \Rightarrow \lim_{n\to\infty} \alpha_i^{(n)} = 0,\ i \in \mathbb{N}.$$

The following observations and questions arise naturally in wake of the above definitions :

(I) Is an admissible sequence to a *minimal sequence unique? One has no for the answer. Indeed, let $\{f_n\} \subset E^*$ be a *minimal sequence with admissible sequence $\{x_n\} \subset E$ such that $\widetilde{[f_n]} \neq E^*$. Then, there is an $x_0 \in E$ such that $f_n(x_0) = 0$, for each n. Put $y_1 = x_1 - x_0$, $y_i = x_{i-1}$, $i = 2, 3, \ldots$. Thus $\{f_n\}$ has another admissible sequence $\{y_n\} \subset E$.

However, an admissible sequence to a *minimal sequence is unique if and only if the *minimal sequence is total.

(II) Every *minimal sequence is both minimal and *ω-linearly independent. Is the converse, in each case, true? We show that each has negative solution (Examples 1 and 4).

(III) Every *Λ-linearly independent sequence is Λ-linearly independent. However, the converse need not be true (Example 2).

(IV) Every normalized *Λ-linearly independent sequence with $\inf\limits_i \lambda_i > 0$ is *ω-linearly independent (Theorem 2), and hence ω-linearly independent. It is shown that the converse may not be true (Example 3).

(V) Does there exist a sequence which is *ω-linearly independent but not Λ-linearly independent and vice-versa? We show that they do (Examples 2 and 3).

(VI) Can we find a sequence which is minimal but not *ω-linearly independent, and conversely? Indeed, we can (Examples 1, 3 and 2).

(VII) Some other questions, namely, "Whether we can find a minimal (ω-linearly independent) sequence which is not *Λ-linearly independent (respectively, *ω-linearly independent)" are intrinsically contained in and answered above.

4. Counter Examples

Example 1—Let $E = c$ and let $\{e_n\}$ be the sequence of unit vectors in E. Define $\{\phi_n\} \subset E^*$ by

$$\phi_n(x) = \alpha_n \; (x = \{\alpha_n\} \in E,\; n \in \mathbb{N}).$$

Then (e_n, ϕ_n) is a biorthogonal system such that $\widetilde{[\phi_n]} = E^*$. Since $[e_n] = c_0 \subset c$, there exists a $\phi \in E^*$ and a $y \in E \setminus c_0$ such that $\phi(e_n) = 0$, $n \in \mathbb{N}$ and $\phi(y) \neq 0$. Then $\{\phi, \phi_n\}$ is a *ω-linearly independent sequence. Indeed, let

$$\alpha_1 \phi(x) + \sum_{i=1}^{\infty} \alpha_{i+1} \phi_i(x) = 0, \text{ for each } x \in E.$$

Then, by taking $x = e_n$, $n \in \mathbb{N}$, one may easily see that $\alpha_i = 0$, $i = 2, 3, \ldots$ and then

by taking $x = y$, $\alpha_1 = 0$. But, since $\phi \in \widetilde{[\phi_n]} = E^*$, $\{\phi, \phi_n\}$ is not *minimal (Theorem 1, (a) $\Leftrightarrow$ (b)).

Example 2—Let $E = l^2$ and let $\{f_n\}$ be the unit vector basis of E. Define $\{g_n\} \subset E$ and $\{\phi_n\} \subset E^*$ by

$$g_n = \tfrac{1}{2} f_1 + \tfrac{1}{2} f_{n+1}, \ (n \in \mathbb{N})$$

and

$$\phi_n(f) = 2\,\xi_{n+1}, \ (f = \{\xi_n\} \in E, \ n \in \mathbb{N}).$$

Then (g_n, ϕ_n) is a biorthogonal system such that $[g_n] = E$. But $\{g_n\}$ is not a weak* basis of E. Therefore, there exists an $f \in E$ which does not admit any expansion of the form

$$f(x) = \sum_{i=1}^{\infty} \alpha_i\, g_i(x)$$

for all x in the predual of E. Then $\{f, g_n\}$ is *ω-linearly independent (see proof of Theorem 3) but not minimal, since $f \in [g_n]$. Moreover, one can write (Singer[4], Theorem 3.1)

$$f(x) = \lim_{n\to\infty} \sum_{i=1}^{m_n} \alpha_i^{(n)}\, g_i(x)$$

for $x \in E$ as above and $\sup\limits_{1<n<\infty} |\alpha_i^{(n)}| < +\infty$, $(i \in \mathbb{N})$.

Then, for $\lambda_1 = 1$ and

$$\lambda_i \geqslant \sup_{1<n<\infty} |\alpha_{i-1}^{(n)}|, \ i = 2, 3, \ldots,$$

it follows that $\{f, g_n\}$ is not *Λ-linearly independent, since for the sequence $\{\beta_i^{(n)}\} \subset \mathbb{K}$ given by

$$\beta_i^{(n)} = \begin{cases} 1 & , \ i = 1, n \in \mathbb{N}, \\ -\alpha_{i-1}^{(n)} & , \ i = 2, 3, \ldots, m_n + 1, n \in \mathbb{N}, \\ 0 & , \ i = m_n + 2, m_n + 3, \ldots \ (n \in \mathbb{N}) \end{cases}$$

we have

$$|\beta_i^{(n)}| \leqslant \lambda_i, \text{ for each } i \text{ and } n, \text{ and}$$

$$\lim_{n\to\infty} \sum_{i=1}^{\infty} \beta_{i+1}^{(n)}\, g_i(x) + f(x) = \lim_{n\to\infty} \Big(f(x) - \sum_{i=1}^{m_n} \alpha_i^{(n)}\, g_i(x) \Big) = 0$$

while

$$\lim_{n\to\infty} \beta_1^{(n)} \neq 0.$$

Example 3—Let $E = c_0$ and let $\{e_n\}$ be the unit vector basis of E^*. Define $\{f_n\} \subset E^*$ by

$$f_1 = e_1,\ f_n = \tfrac{1}{2}e_{n-1} - \tfrac{1}{2}e_n\ (n = 2, 3, \ldots).$$

Then $\{f_n\} \subset E^*$ is a normalized minimal sequence[10]. Since

$$\tfrac{1}{2}f_1(x) - \sum_{i=2}^{\infty} f_i(x) = 0, \text{ for all } x \in E,$$ $\{f_n\}$ is not $^*\omega$-linearly independent.

Example 4—Let $E = c_0$, $\{e_n\}$ be the unit vector basis of E^* and $\{\phi_n\}$ the sequence of unit vectors in E^{**}. Define $\{f_n\} \subset E^*$ and $\{\psi_n\} \subset E^{**}$ by

$$f_1 = (1, 0, 0, \ldots)$$

$$f_n = ((-1)^{n+1}, 0, 0, \ldots, \underset{\substack{\downarrow \\ nth \text{ place}}}{1}, 0, \ldots),\ (n \geqslant 2),$$

$$\psi_1 = (1, 1, -1, 1, -1, 1, \ldots)$$

and

$$\psi_n = (0, 0, \ldots, \underset{\substack{\downarrow \\ nth \text{ place.}}}{1}, 0, \ldots),\ (n = 2, 3, \ldots).$$

Then, (f_n, ψ_n) is a biorthogonal system and hence $\{f_n\}$ is minimal[12]. Since $\psi_1 \notin \pi(E)$, ψ_1 is not weak*-continuous. Also, since $[f_n] = E^*$, $\{\psi_n\}$ is the only sequence such that (f_n, ψ_n) is a biorthogonal system. Hence $\{f_n\}$ is not *minimal.

5. Main Results

Theorem 1—Let $\{f_n\} \subset E^*$ be a sequence. The following statements are equivalent :

(a) $\{f_n\}$ is *minimal.

(b) $f_n \notin \widetilde{[f_i]}_{i\neq n}$, $(i, n \in \mathbb{N})$.

Each of these statements imply the following :

(c) $\{f_n\}$ is $^*\Lambda$-linearly independent.

(d) $\{f_n\}$ is $^*\omega$-linearly independent.

The statements (a) and (b) (hence (c) and (d)) are implied by each of the following:

(e) There exists $\{u_n\} \subset L(E, E)$ such that

$$u_n^*(f) = f, \quad f \in [f_i]_{i=1}^n$$

and

$$u_n^*(f) = 0, \quad f \in [f_i]_{i=n+1}^\infty .$$

(f) There exists a sequence $\{x_n\} \subset E$ such that

$$\lim_{n\to\infty} \sum_{i=1}^{m_n} \alpha_i^{(n)} f_i = f.$$

$$\Rightarrow \lim_{n\to\infty} \alpha_i^{(n)} = f(x_i) \ (i \in \mathbb{N}).$$

PROOF : (a) $\Rightarrow$ (b) -- Let $\{x_n\} \subset E$ be a sequence in E such that $f_i(x_j) = \delta_{ij}$ $(i, j \in \mathbb{N})$. Suppose that $f_n \in \widetilde{[f_i]}_{i\neq n}$. Then, for each n, $g_n = f_n \mid [x_n]$ is in the $\sigma([x_n]^*, [x_n])$-closure of $\{g_i\}_{i\neq n}$ in $[x_n]^*$. Since $[x_n]$ is separable, the $\sigma([x_n]^*; [x_n])$-topology is metrizable on bounded sets of $[x_n]^*$ (Wilansky[11], Theorem 3.1.1). Therefore, each g_n in the $\sigma([x_n]^*, [x_n])$-closure of $\{g_i\}_{i\neq n}$ has the form

$$g_n = \sigma([x_n]^*, [x_n]) - \lim_{k\to\infty} \sum_{\substack{i=1\\ i\neq n}}^{m_k} \alpha_i^{(k)} g_i.$$

Then

$$1 = f_n(x_n) = g_n(x_n) = 0.$$

This is not possible.

$(b) \Rightarrow (a)$—Let $f_n \notin \widetilde{[f_i]}_{i\neq n}$ for each i and n. Then, there exists a sequence $\{x_n\} \subset E$ such that $f_n(x_n) = 1$ and $f(x_n) = 0$, for each $f \in \widetilde{[f_i]}_{i\neq n}$. Hence, $\{f_n\}$ is a *minimal sequence in E^*.

$(a) \Rightarrow (c)$ – Let $\{x_n\} \subset E$ be a sequence such that $f_i(x_j) = \delta_{ij}$ $(i, j \in \mathbb{N})$. Let $|\alpha_i^{(n)}| \leqslant \lambda_i$ $(i \in \mathbb{N})$ and

$$\lim_{n\to\infty} \sum_{i=1}^{\infty} \alpha_i^{(n)} f_i(x) = 0$$

for every $x \in E$. Then

$$\lim_{n\to\infty} \alpha_k^{(n)} = \lim_{n\to\infty} \sum_{i=1}^{\infty} \alpha_i^{(n)} f_i(x_k) = 0, \ (k \in \mathbb{N}).$$

$(a) \Rightarrow (d)$—Obvious.

$(e) \Rightarrow (a)$— $u_1^*(f_1) = f_1$. Let y_1 be in the range of u_1 and $x_1 \in E$ such that $y_1 = u_1(x_1)$ and $f_1(y_1) = 1$.

Then

$$f_i(y_1) = f_i(u_1(x_1))$$

$$= (u_1^*(f_i))(x_1)$$

$$= 0, \ i = 2, 3, \ldots.$$

Again, let y_2 be in the range of $u_2 - u_1$ and $x_2 \in E$ such that $y_2 = (u_2 - u_1)(x_2)$ and $f_2(y_2) = 1$. Then

$$f_1(y_2) = f_1((u_2 - u_1)(x_2))$$

$$= (u_2^*(f_1))(x_2) - (u_1^*(f_1))(x_2) = 0$$

and

$$f_i(y_2) = (u_2^*(f_i))(x_2) - (u_1^*(f_i))(x_2)$$

$$= 0, \ i = 3, 4, \ldots.$$

Continuing like this, let y_n be in the range of $u_n - u_{n-1}$ and $x_n \in E$ such that $y_n = (u_n - u_{n-1})(x_n)$ and $f_n(y_n) = 1$. Then for $i \neq n \in \mathbb{N}$, $f_i(y_n) = 0$. Hence, $\{f_n\}$ is a *minimal sequence in E^*.

$(f) \Rightarrow (a)$—Define

$$\phi_j\left(\sum_{i=1}^{n} \alpha_i f_i\right) = \alpha_j, \left(\sum_{i=1}^{n} \alpha_i f_i \in \text{span}\{f_n\}, j = 1, 2, \ldots, n\right).$$

Note that ϕ_j is a well defined continuous linear functional on span $\{f_n\}$ since, by the assumption in (f), $\{f_n\}$ is finitely linearly independent (i. e. every finite subsequence of $\{f_n\}$ is linearly independent). Further, since any $f \in [f_n]$ has the form

$$f = \lim_{n\to\infty} \sum_{i=1}^{m_n} \alpha_i^{(n)} f_i$$

one can extend ϕ_j to $[f_n]$ by writing

$$\phi_j(f) = \lim_{n\to\infty} \alpha_j^{(n)} = f(x_j), \ (j \in \mathbb{N}).$$

Hence

$$\phi_j(f_i) = f_i(x_j) = \delta_{ij}, \ (i, j \in \mathbb{N}).$$

Whence $\{f_n\}$ is *minimal.

Theorem 2—Let $\Lambda = \{\lambda_i\}$ with $\lambda_i > 0$, $i \in \mathbb{N}$. If $\inf_i \lambda_i > 0$, then every normalized *Λ-linearly independent sequence in E^* is *ω-linearly independent (hence ω-linearly independent).

Proof : Let $\{f_n\} \subset E^*$ be a normalized *Λ-linearly independent sequence. Assume that $\{f_n\}$ is not *ω-linearly independent. Then, there is a sequence $\{\alpha_n\} \subset \mathbb{K}$ with $\sup_n |\alpha_n| \neq 0$ such that

$$\sum_{i=1}^{\infty} \alpha_i \; f_i(x) = 0$$

for all $x \in E$. Since for each $x \in E$,

$$\sup_n |\alpha_n \, f_n(x)| < \infty$$

the uniform boundedness principle shows that

$$\sup_n |\alpha_n| = \sup_n \|\alpha_n f_n\| < \infty.$$

Put $\gamma = \inf_i \lambda_i / \sup_i |\alpha_i|$ and $\alpha_i^{(n)} = \gamma\alpha_i$, $(i, n \in \mathbb{N})$. Then

$$\begin{aligned} |\alpha_i^{(n)}| &= |\gamma\alpha_i| \\ &\leqslant (\inf_i \lambda_i / \sup_i |\alpha_i|)\, |\alpha_i| \\ &\leqslant \lambda_i \; (i, n \in \mathbb{N}) \end{aligned}$$

and

$$\lim_{n\to\infty} \sum_{i=1}^{\infty} \alpha_i^{(n)} f_i(x) = \gamma \sum_{i=1}^{\infty} \alpha_i f_i(x) = 0$$

for all $x \in E$. Since $\{f_n\}$ is *Λ-linearly independent, it follows that $\alpha_i = 0$, for each i. This completes the proof.

Definition 4—A *ω-linearly independent sequence $\{f_n\} \subset E^*$ is said to be maximal if there exists no sequence $\{g_n\} \subset E^*$ with $\{f_n\} \subset \{g_n\}$ such that $\{g_n\}$ is *ω-linearly independent.

Theorem 3—Let $\{f_n\} \subset E^*$ be a *ω-linearly independent sequence. Then $\{f_n\}$ is a weak*-basis of E^* if and only if $\{f_n\}$ is a maximal *ω-linearly independent sequence.

PROOF : Let $\{f_n\}$ be not a weak*-basis of E^*. Then, there is an $f \in E^*$ which does not admit any expansion of the form

$$f(x) = \sum_{i=1}^{\infty} \alpha_i f_i(x) \qquad \ldots(5.1)$$

for each $x \in E$ and any sequence $\{\alpha_n\}$ of scalars. Therefore, $\{f, f_n\}$ is a *ω-linearly independent sequence in E^*. Indeed, let

$$\alpha_1 f(x) + \sum_{i=1}^{\infty} \alpha_{i+1} f_i(x) = 0$$

for every $x \in E$.

If $\alpha_1 = 0$, then $\alpha_{i+1} = 0$, $i \in \mathbb{N}$, since $\{f_n\}$ is a *ω-linearly independent sequence. If $\alpha_1 \neq 0$, then

$$f(x) = \sum_{i=1}^{\infty} \beta_i f_i(x)$$

for every $x \in E$ and $\beta_i = \alpha_{i+1}/\alpha_1$. This is a contradication. Hence, $\{f_n\} \subset E^*$ is not maximal.

Conversely, let $\{f, f_n\} \subset E^*$ be an extension of the *ω-linearly independent sequence $\{f_n\} \subset E^*$. Then, for any sequence $\{\alpha_n\}$ of scalars, f does not admit any expansion of the form (5.1), for each $x \in E$. Because if it does, then

$$f(x) + \sum_{i=1}^{\infty} (-\alpha_i) f_i(x) = 0$$

for every $x \in E$. Whence, $\{f, f_n\} \subset E^*$ is not a *ω-linearly independent sequence. This is a contradiction.

Remark : There do exist maximal *ω-linearly independent sequences in conjugate Banach spaces (Example 4). There also exists a complete *ω-linearly independent sequence which is not maximal (Example 2).

The final picture that emerges from our discussion so far, regarding a sequence in a conjugate Banach space, is given in the following diagram of implications and counter-implications (Figure 1).

6. Deficient *Minimnl Sequences

Definition 5—A sequence $\{f_n\} \subset E^*$ is said to be a deficient *minimal if it can be transformed into a *minimal sequence by omitting a finite number of its elements.

Towards the existence of deficient *minimal sequences, one can see that the sequence $\{\phi, \phi_n\} \subset E^*$ in Example 1 and the sequence $\{f_n\} \subset E^*$ in Example 4 are deficient *minimal sequences.

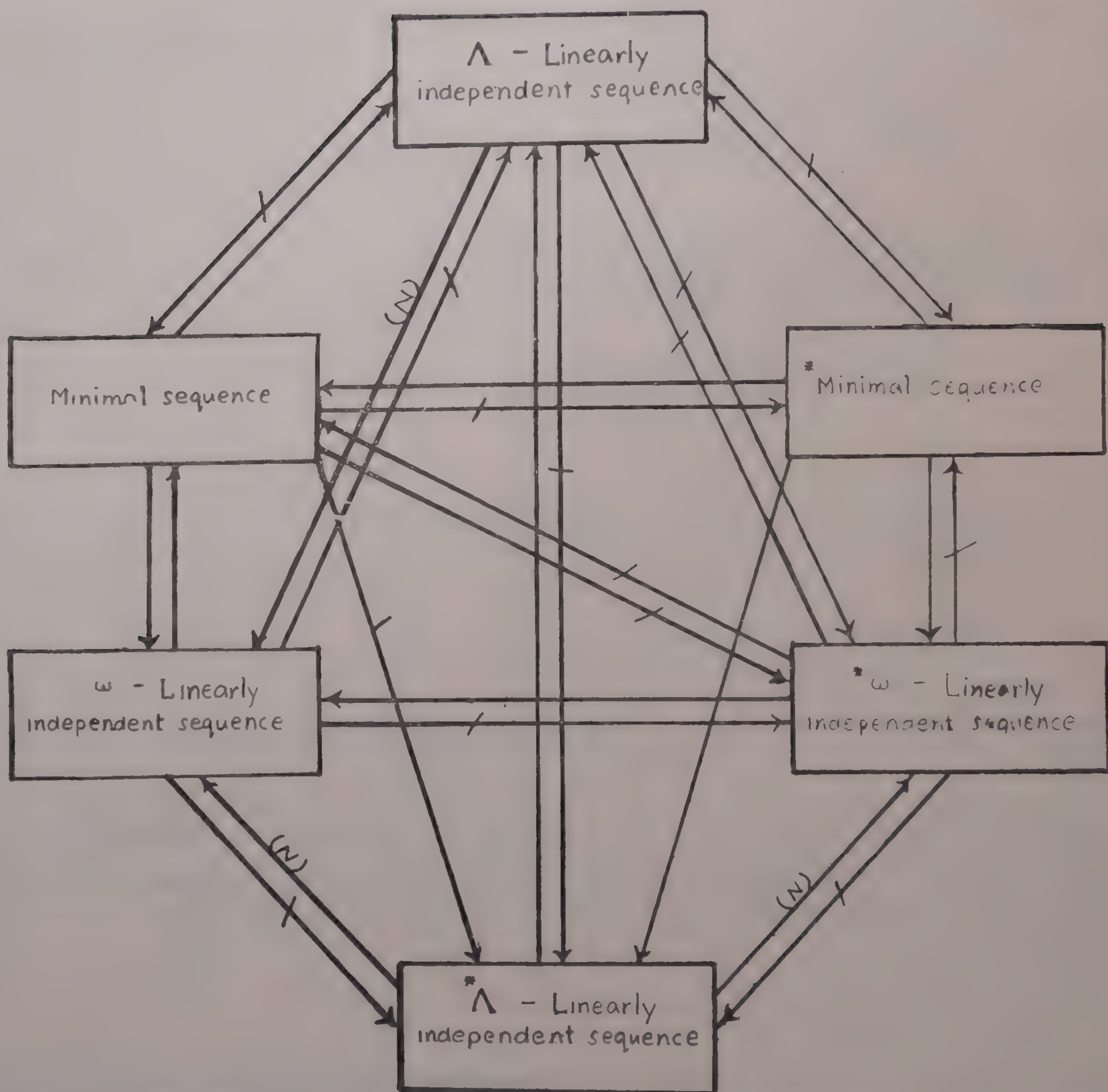

FIG. 1. (N) These imlications are valid for normalized sequences such that $\inf_i \lambda_i > 0$.

Theorem 4—Let E be separable and $\{f_n\} \subset E^*$ be such that for every infinite set of positive integers $\{\sigma(k)\}_{k=1}^{\infty}$

$$\widetilde{[f_i]}_{i \neq \sigma(1), \sigma(2), \ldots} \neq \widetilde{[f_i]}_{i=1}^{\infty} \quad .$$

Then $\{f_n\} \subset E^*$ is a deficient *minimal seqnence.

PROOF : Assume that $\{f_n\} \subset E^*$ is not a deficient *minimal sequence. Then, in view of Theorem 1, there exists an index $\sigma(1)$ such that

$$f_{\sigma(1)} \in \widetilde{[f_i]}_{i \neq \sigma(1)} = \widetilde{[f_i]}_{i=1}^{\infty}$$

and $\| f_{\sigma(1)} \| = 1$. Since E is separable, the unit ball U of E^* is metrizable in the w*-

topology. Let d be a metric on U. Since $f_{\sigma(1)} \in U \subset [\widetilde{f_i}]_{i=1}^{\infty}$, there exists an index $\alpha(1) > \sigma(1)$ and scalars $\gamma_1^{(1)}, \gamma_2^{(1)}, \ldots, \gamma_{\alpha(1)}^{(1)}$ such that

$$\sum_{j=1}^{\alpha(1)} \gamma_i^{(1)} f_i \in U$$

and

$$d\,(f_{\sigma(1)}, \sum_{i=1}^{\alpha(1)} \gamma_i^{(1)} f_i) < \tfrac{1}{2}.$$

By assumption, $\{f_i\}_{i=\alpha(1)+1}^{\infty}$ is not *minimal. Again, by Theorem 1, there exists an index $\sigma(2) \geqslant \alpha(1) + 1$ such that

$$f_{\sigma(2)} \in [\widetilde{f_i}]_{\substack{i=\alpha(1)+1 \\ i\neq\sigma(2)}}^{\infty} = [\widetilde{f_i}]_{i=\alpha(1)+1}^{\infty}$$

and $\| f_{\sigma(2)} \| = 1$. Then, $f_{\sigma(2)} \in U$. Thus, there exists an index $\alpha(2) \geqslant \sigma(2)$ and scalars $\gamma^{(2)}, 1, \ldots, \gamma_{\alpha(2)}^{(2)}$ such that

$$\sum_{i=1}^{\alpha(2)} \gamma_i^{(2)} f_i \in U$$

and

$$d\,(f_{\sigma(2)}, \sum_{\substack{i=1 \\ i\neq\sigma(1),\sigma(2)}}^{\alpha(2)} \gamma_i^{(2)} f_i) < \tfrac{1}{4}.$$

Since $[\widetilde{f_i}]_{i\neq\sigma(1),\,\sigma(2)} = [\widetilde{f_i}]_{i=1}^{\infty}$, one can choose the index $\alpha(2)$ such that we also have

$$d\,(f_{\sigma(1)}) \sum_{\substack{i=1 \\ i\neq\sigma(1),\sigma(2)}}^{\alpha(2)} \gamma_i^{(2)} f_i) < \tfrac{1}{4}.$$

Proceeding like this we get two infinite sets of positive integers $\{\sigma(n)\}_{n=1}^{\infty}$ and $\{\alpha(n)\}_{n=1}^{\infty}$ with $\alpha(n-1) + 1 \leqslant \sigma(n) \leqslant \alpha(n)$, $\alpha(0) = 0$ and scalars $\gamma_i^{(n)} (i, j \in \mathbb{N})$,

such that

$$\sum_{i=1}^{\alpha(n)} \gamma_i^{(n)} f_i \in U$$

and

$$d\,(f_{\sigma(k)}, \sum_{\substack{i=1 \\ i\neq\sigma(1),\sigma(2),\cdots,\sigma(n)}}^{\alpha(n)} \gamma_i^{(n)} f_i) < \frac{1}{2^n} (k = 1, 2, ..., n, n \in \mathbb{N}).$$

Thus,

$$f_{\sigma(k)} \in [\widetilde{f_i}]_{i\neq\sigma(1),\sigma(2),\cdots,} k \in \mathbb{N}.$$

Hence,

$$[\widetilde{f_i}]_{i=1}^{\infty} = [\widetilde{f_i}]_{i\neq\sigma(1),\alpha(2),\cdots}.$$

Theorem 5—Let E be separable and let $\{f_n\} \subset E^*$ be a sequence which is not deficient *minimal. Then, for every sequence of positive integers $\{C_n\}$, there exists a total sequence $\{g_n\} \subset E^*$ such that

$$\| f_n - g_n \| \leqslant C_n \ (n \in \mathbb{N}).$$

PROOF : Since $\{f_n\}$ is not deficient *minimal, there exists, by Theorem 4, an infinite sequence of indices $\{\sigma(n)\}_{n=1}^{\infty}$ such that

$$[\widetilde{f_{\alpha(n)}}]_{n=1}^{\infty} = [\widetilde{f_i}]_{i\neq\sigma(1),\sigma(2),\cdots} = [\widetilde{f_i}]_{i=1}^{\infty} \qquad ...(5.2)$$

where $\{\alpha(n)\}_{n=1}^{\infty} = \mathbb{N} \setminus \{\sigma(n)\}_{n=1}^{\infty}$. Let $\{h_n\} \subset E^*$ be a total sequence such that $\|h_n\| \leqslant 1$, $n \in \mathbb{N}$.

Put

$$g_{\sigma(i)} = f_{\sigma(i)} + C_{\sigma(i)} h_i \ (i \in \mathbb{N}) \qquad ...(5.3)$$

and

$$g_{\alpha(i)} = f_{\alpha(i)} \ (i \in \mathbb{N}) \qquad ...(5.4)$$

where $\{C_n\}$ is any sequence of positive integers. Then

$$\| f_i - g_i \| \leqslant C_i \ (i \in \mathbb{N})$$

and $[\widetilde{g_n}] = E^*$. Indeed, if $[\widetilde{g_n}] \neq E^*$, then there exists a $\phi \in E^*$ and $x_0 \in E$ such that $\phi\,(x_0) = 1$ and $g_n\ (x_0) = 0$ for each n. Thus, by (5.2) and (5.4), is follows that

$f_n\ (x_0) = 0$ for each n. Since $C_{\sigma(i)} \neq 0$ for each i, it follows from (5.3) that $h_i\ (x_0) = 0$. Since $\phi\ (x_0) = 1$, we have $\widetilde{[h_n]} \neq E^*$.

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ASYMPTOTIC BEHAVIOUR OF SOLUTIONS OF FUNCTIONAL DIFFERENTIAL EQUATIONS

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This paper studies the equation

$$L_n\, x(t) + f(t, x(t), x(g(t))) = h(t), \ t > 0, \ n \geqslant 2$$

where the differential operator L_n is defined by

$$L_0\, x(t) = x(t),$$

$$L_i\, x(t) = \frac{1}{r_i(t)} \frac{d}{dt} L_{i-1}\, x(t) \ i = 1, 2, \ldots, n; r_n(t) = 1$$

and a necessary and sufficient condition that all oscillatory solutions of the above equation converge to zero asymptotically is presented. The results obtained extend and improve some of the previous known results.

1. INTRODUCTION

In the last few years the oscillatory and non-oscillatory behaviour of solutions of both ordinary and functional differential equations of arbitrary order have received a great deal of attention. For example see reference[2-10] and the reference therein.

Recently Chen and Yeh[3] have established necessary and sufficient conditions for the asymptotic decay of oscillations in delayed functional equations of the form

$$L_n\, x(t) + f(t, x(g(t))) = h(t), \ t > 0, \ n \geqslant 2$$

where the differential operator L_n is defined by

$$L_0\, x(t) = \frac{x(t)}{r_0(t)},$$

$$L_i\, x(t) = \frac{1}{r_i(t)} \frac{d}{dt} L_{i-1}\, x(t), \ i = 1, \ldots, n; \ r_n(t) = 1.$$

In this paper we shall establish necessary and sufficient conditions for the oscillatory and asymptotic behaviour of the functional differential equation of the form

$$L_n x(t) + f(t, x(t), x(g(t))) = h(t), \ t > 0, \ n \geqslant 2 \qquad \ldots(1)$$

where the differential operator L_n is defined by

$$L_0 x(t) = x(t),$$

$$L_i x(t) = \frac{1}{r_i(t)} \frac{d}{dt} L_{i-1} x(t), \ i = 1, 2, \ldots, n; r_n(t) = 1.$$

The results obtained here generalize and improve some of the known results[3-10].

We consider the differential equation (1) together with the following conditions : (i) $f \in [R_+ \times R^2, R]$ and there exist positive functions $P(t)$, $q(t)$ and $H_i(t) \in C[R_+, R_+]$ $(i = 1, 2)$ with $H_i(t)$ non-decreasing and $H_i(Kt) \leqslant H_i(K) H_i(t)$ for any $K > 0$ such that

$$f(t, u, v) \leqslant p(t) H_1(|u|) + q(t) H_2(|v|)$$

and

$$\int_{r_0}^{r} \frac{ds}{H_i(s)} \to \infty \quad \text{as} \quad r \to \infty, \ r \geqslant r_0 > 0.$$

(ii) $g, h \in C[R_+, R_+]$, $g(t) \leqslant t$, $\lim\limits_{t \to \infty} g(t) = \infty$.

(iii) $\limsup\limits_{t \to \infty} \dfrac{w_i(t, u)}{w_{n-1}(t, u)} < \infty$, $i = 1, 2, \ldots, n-2$, for each fixed $u > 0$,

where $w_i(t, u)$ is defined by

$$w_i(t, u) = \int_u^t r_1(s_1) \int_u^{s_1} r_2(s_2) \ldots \int_u^{s_{i-1}} r_i(s_i)\, ds_i \ldots ds_1.$$

In what follows the term "solution" is used only for such solutions of (1) which are defined for all large t. The oscillatory character is considered in the usual sense, that is, a continuous real valued function which is defined on an interval of the form $[t_0, \infty)$ is called oscillatory if it has no last zero and is called non-oscillatory otherwise.

2. Main Results

Theorem 1—Suppose that for any $T \geq 0$

$$\int_T^\infty p(t) H_1(w_{n-1}(t, T))\, dt < \infty, \ \int_T^\infty q(t) H_2(w_{n-1}(t, T))\, dt < \infty \qquad \ldots(2)$$

and

$$\int_T^\infty h(t)\,dt < \infty. \quad \ldots(3)$$

Let $x(t)$ be a solution of (1). Then $(x(g(t)) = O(w_{n-1}(t, T))$ as $t \to \infty$.

PROOF : Let $x(t)$ be a solution of (1) on an interval $[t_0, \infty)$, $t_0 \geqslant 0$. It follows from (ii) and (iii) that there exists a $T \geqslant t_0$ and a positive constant C such that

$$g(t) \geqslant t_0 \text{ for } t \geqslant T$$

and

$$w_i(t, T) \leqslant C\, w_{n-1}(t, T),\ i = 1, 2, \ldots, n-2.$$

Now a simple argument shows that

$$|x(t)| \leqslant \sum_{i=0}^{n-1} w_i(t, T)\, |L_i x(T)| + \int_T^t w_{n-1}(s, T)\, |L_n x(s)|\, ds$$

$$\frac{|x(t)|}{w_{n-1}(t; T)} \leqslant C \sum_{i=0}^{n-1} |L_i x(T)| + \int_T^t |h(s)|\, ds$$

$$+ \int_T^t [p(s) H_1(|x(s)|) + q(s) H_2(|x(g(s))|]\, ds$$

$$\leqslant M + \int_T^t [p(s) H_1(|x(s)|)$$

$$+ q(s) H_2(|x(g(s))|)]\, ds \quad \ldots(4)$$

where

$$M = C \sum_{i=0}^{n-1} |L_i x(T)| + \int_T^\infty |h(s)|\, ds.$$

From (4), we have

$$\frac{|x(t)|}{w_{n-1}(t, T)} \leqslant F(t),\ t > T$$

where

$$F(t) = M + \int_T^t [p(s) H_1(|x(s)|) + q(s) H_2(|x(g(s))|)]\, ds. \quad \ldots(5)$$

Then

$$|x(t)| \leqslant w_{n-1}(t, T)\, F(t)$$

and

$$| x (g (t)) | \leqslant w_{n-1} (t, T) F (t)$$

for $t > T$, since $w_{n-1} (t, T)$ and $F (t)$ are non-decreasing. Hence from (5) we have

$$F (t) \leqslant M + \int_T^t [p (s) H_1 (w_{n-1} (s, T)) H_1 (F (s))$$
$$+ q (s) H_2 (w_{n-1} (s, T)) H_2 (F (s))] \, ds.$$

By an application of Theorem 11 in[1], we obtain

$$F (t) \leqslant D_1 (t) D_2 (t), \; t > T$$

where

$$D_1 (t) = G_1^{-1} \left[G_1 M + \int_T^t p (s) H_1 (w_{n-1} (s, T)) \, ds] \right]$$

$$D_2 (t) = G_2^{-1} \left[G_2 (M) + \int_T^t q (s) H_2 (w_{n-1} (s, T)) D_1 (s) \, ds \right]$$

and $G_i (r) = \int_{r_0}^{r} \frac{ds}{H_i (s)}$ and G_i^{-1} is the inverse of G_i , $(i = 1, 2)$.

This and (2) imply that $\frac{| x (g (t)) |}{w_{n-1} (t, T)}$ is bounded. This completes the proof.

Remark : For $H_2 (u) = | u |^r$, $r \in (0, 1]$, Theorem 1 generalizes and improves the results of Chen and Yeh[3] (Theorem 1), Chen *et al.*[2] (Theorem 1), Singh and Kusano[7] (Theorem 1) and Tang[10] (Theorem 1) which require the condition

$$\int^{\infty} r_i (t) \, dt = \infty \quad \text{for} \quad i = 1, 2, \ldots, n - 1.$$

Using Theorem 1 we can prove the following theorem which extends Theorem 2 of Chen and Yeh[3] and Theorem 3 of Philos[8].

Theorem 2—Let (2) and (3) hold. Assme that for some $T \geqslant 0$

$$\int_T^{\infty} r_1 (s_1) \int_{s_1}^{\infty} r_2 (s_2) \ldots \int_{s_{n-1}}^{\infty} [p (s) H_1 (w_{n-1} (s, T))$$
$$+ q (s) H_2 (w_{n-1} (s, T))] \, ds \, ds_{n-1} \ldots ds_1 < \infty \qquad \ldots(6)$$

$$\int_T^{\infty} r_1 (s_1) \int_{s_1}^{\infty} r_2 (s_2) \ldots \int_{s_{n-1}}^{\infty} | h (s) | \, ds \, ds_{n-1} \ldots ds_1 < \infty. \qquad \ldots(7)$$

Let $x(t)$ be an oscillatory solution of (1). Then

$$\lim_{t\to\infty} L_i x(t) = 0 \quad \text{for} \quad i = 0, 1, \ldots, n-1.$$

PROOF : The proof of Theorem 2 can be modelled as that of Theorem 3 in Philos[8] and hence is omitted.

Example 1—Consider the differential equation

$$x'' + e^{-t-\sin t} x(t - \sin t) = e^{-2t} \sin(t - \sin t) - 2e^{-t} \cos t, \qquad 0 \leqslant t \leqslant \pi. \qquad \ldots(E_1)$$

All conditions of Theorem 2 are satisfied. It has $x(t) = e^{-t} \sin t$ as an oscillatory solution which approaches zero as $t \to \infty$.

Example 2—Consider the differential equation

$$(e^{-t}(e^t x')')' + e^{-t} x^2(t) = 2e^{-t} \sin t + e^{-3t} \sin^2 t. \qquad \ldots(E_2)$$

All the conditions of Theorem 2 are satisfied. It has $x(t) = e^{-t} \sin t$ as an oscillatory solution which approaches zero as $t \to \infty$.

Remark 2 : Since $r_1(t) = e^{-t} > 0$ implies $r'(t) = -e^{-t} < 0$, the condition $r'(t) \geqslant 0$ of Singh[9] is not satisfied. Also $\int^{\infty} e^{-t}\, dt$ does not tend to ∞, and so condition $\int^{\infty} r_1(t)\, dt = \infty$ of Chen and Yeh[3] is not satisfied.

Example 3—The differential equation

$$x'' + \frac{1}{t^2} x(t) + \frac{1}{t^2} x(t - e^{-t})$$
$$= \frac{1}{t^2} [\sin(\log(t - e^{-t})) - \cos(\log t)] \qquad \ldots(E_3)$$

has the oscillatory solution $x(t) = \sin(\log t)$ but $\lim_{t\to\infty} x(t)$ does not exist. Note that only condition (7) is violated.

Example 4—Consider the differential equation

$$(e^{-t} x')' + e^{-3t} x^2(t) + e^{-4t-6\pi} x^3(t - \pi) = 5e^{-3t}(\sin t - \cos t)$$
$$+ e^{-7t} \sin^2 t - e^{-10t} \sin^3 t. \qquad \ldots(E_4)$$

All conditions of Theorem 2 are satisfied. It has $x(t) = e^{-2t} \sin t$ as an oscillatory solutions which approaches zero as $t \to \infty$.

Example 5—Consider the differential equation

$$(e^{-t} x'(t))'' + (e^{-2t} + 2e^{-t}) x(t) + e^{-2t-\pi} x(t - \pi)$$
$$= 7e^{-2t} \cos t + 3e^{-2} \sin t. \qquad \ldots(E_5)$$

All the conditions of Theorem 2 are satisfied. It has $x(t) = e^{-t} \sin t$ as an oscillatory solution which approaches zero as $t \to \infty$.

Example 6—Consider the differential equation

$$x^{(4)}(t) + t^{-(14/3)} \sqrt[3]{x(t)} = \frac{1}{ct^5} [-40 \cos(\log t) - 10 \sin(\log t) + \sqrt[3]{\sin(\log t)}]. \quad \ldots(E_6)$$

All the conditions of Theorem 2 are satisfied. It has an oscillatory solution $x(t) = \frac{1}{t} \sin(\log t)$ which approaches zero as $t \to \infty$.

Next we prove a theorem which extends Theorem 3 of Chen and Yeh[3] and Theorem 8 of Grace and Lalli[4].

Theorem 3—Let conditions (2), (3) and (6) hold. Suppose that

$$\lim_{t\to\infty} \frac{h(t)}{p(s) H_1(w_{n-1}(t, T)) + q(t) H_2(w_{n-1}(t, T))}$$

exists. Then a necessary and sufficient condition for all oscillatory solution of (1) to approach zero asymptotically is that

$$\lim_{t\to\infty} \frac{h(t)}{p(s) H_1(w_{n-1}(t, T)) + q(s) H_2(w_{n-1}(t, T))} = 0. \quad \ldots(8)$$

Proof : Sufficiency—Suppose (8) holds. Then

$$|h(t)| < p(t) H_1(w_{n-1}(t, T)) + q(t) H_2(w_{n-1}(t, T))$$

for t large enough. Since (6) holds, we have (7) and the conclusion follows from Theorem 2.

Necessity—Let $x(t)$ be an oscillatory solution of (1) approaching zero as $t \to \infty$. Suppose that,

$$\frac{|h(t)|}{p(t) H_1(w_{n-1}(t, T)) + q(t) H_2(w_{n-1}(t, T))} \geqslant m > 0.$$

Dividing equation (1) by

$$p(t) H_1(w_{n-1}(t, T)) + q(t) H_2(w_{n-1}(t, T))$$

we have

$$\frac{L_n x(t)}{p(t) H_1(w_{n-1}(t, T)) + q(t) H_2(w_{n-1}(t, T))} + \frac{f(t, x(t), x(g(t)))}{p(t) H_1(w_{n-1}(t, T)) + q(t) H_2(w_{n-1}(t, T))}$$

(*equation continued on p.* 1102)

$$= \frac{h(t)}{p(t)\, H_1(w_{n-1}(t, T)) + H_2(w_{n-1}(t, T))}\,.$$

Hence we get for t large enough

$$\frac{L_n\, x(t)}{p(t)\, H_1(w_{n-1}(t, T)) + q(t)\, H_2(w_{n-1}(t, T))} + \frac{|\, x(t)\, |}{H_1(w_{n-1}(t, T))} + \frac{|\, x(g(t))\, |}{H_2(w_{n-1}(t, T))}$$
$$\geqslant \frac{h(t)}{p(t)\, H_1(w_{n-1}(t, T)) + q(t)\, H_2(w_{n-2}(t, T))} > 0$$

or

$$\frac{L_n\, x(t)}{p(t)\, H_1(w_{n-1}(t, T)) + q(t)\, H_2(w_{n-1}(t, T))} - \frac{|\, x(t)\, |}{H_1(w_{n-1}(t, T))} - \frac{|\, x(g(t))\, |}{H_2(w_{n-1}(t, T))}$$
$$\leqslant \frac{h(t)}{p(t)\, H_1(w_{n-1}(t, T)) + q(t)\, H_2(w_{n-1}(t, T))} < 0.$$

Taking the limit as $t \to \infty$, we find that $L_n\, x(t)$ has one sign for sufficiently large t. This fact and (iii) restrict $x(t)$ to eventually assume a constant sign, which is a contradiction.

Remark 3 : In Example 5, put $h(t) = 7e^{-2t} \cos t + 3e^{-2t} \sin t$, $p(t) = e^{-2t} + 2e^{-t}$, $q(t) = e^{-2t-\pi}$, $w_2(t, T) = e^t$ and $H_1(w_2(t, T)) = H_2(w_2(t, T)) = e^t$, then

$$\lim_{t \to \infty} \frac{h(t)}{p(t)\, H_1(w_2(t, T)) + q(t)\, H_2(w_2(t, T))} = 0.$$

All conditions of Theorem 3 are satisfied. Hence all oscillatory solutions of (E_5) approach zero as $t \to \infty$.

Theorem 4—Let (2), (3) and (7) hold. If

$$\frac{h(t)}{p(t)\, H_1(w_{n-1}(t, T)) + q(t)\, H_2(w_{n-1}(t, T))}$$

is bounded, then all oscillatory solutions of (1) tend to zero as $t \to \infty$.

PROOF : The proof follows from Theorem 2 and the sufficiency part of Theorem 3.

Example 7—Consider the differential equation

$$(e^{-t}\, x'(t))' + 4t^2\, e^{-t}\, x(t) + t^2\, e^{-t-\pi}\, x(t - \pi)$$
$$= e^{-2t}\, [2 \sin t^2 - 6t \cos t^2 + 2 \cos t^2 + t^2 \sin (t - \pi)^2] \quad .. (E_7)$$

which has $x(t) = e^{-t} \sin t^2$ as an oscillatory solution. We notice that

$$\frac{h(t)}{p(t) H_1(w_2(t, T)) + q(t) H_2(w_2(t, T))}$$

is bounded and does not tend to a limit as $t \to \infty$. The equation (E_7) satisfies all the conditions and conclusions of Theorem 4 but not that of Theorem 3.

Theorem 5—Suppose that

$$\lim_{t\to\infty} \inf \frac{|h(t)|}{p(t) + q(t)} > 0.$$

Let $x(t)$ be an oscillatory solution of (1). Then

$$\lim_{t\to\infty} \sup |x(t)| > 0.$$

Theorem 6—Let (2), (3) and (7) hold, of

$$\lim_{t\to\infty} \inf \frac{|h(t)|}{p(t) H_1(w_{n-1}(t, T)) + q(t) H_2(w_{n-1}(t, T))} > 0$$

and

$$\frac{h(t)}{p(t) H_1(w_{n-1}(t, T)) + q(t) H_2(w_{n-1}(t, T))}$$

is bounded, then all solutions of eqn. (1) are non-oscillatory.

The proofs of Theorems 5 and 6 are essentially the same as that of Theosems 5 and 6 in Chen and Yeh[3], so we omit the details.

Theorem 7—Let conditions (6) and (7) hold. If there exists a $c > 0$ such that either

$$\lim_{t\to\infty} \inf \int_T^t [h(s) - (p(s) H_1(c) + q(s) H_2(c)] \, ds > 0 \qquad \ldots(9)$$

or

$$\lim_{t\to\infty} \sup \int_T^t [h(s) + (p(s) H_1(c) + q(s) H_2(c))] \, ds < 0 \qquad \ldots(10)$$

for all large T, then any solution $x(t)$ of (1) satisfying $x(t) = O(w_{n-1}(t, T))$ as $t \to \infty$ is non-oscillatory.

PROOF : Let $x(t)$ be an oscillatory solution of (1) satisfying $x(t) = O(w_{n-1}(t, T))$. Then by Theorem 2, $L_k x(t) \to 0$ as $t \to \infty$, $k = 0, 1, 2, \ldots, n-1$. Thus there exists $T \geqslant t_0$ such that

$$L_{n-1} x(T) = 0, \quad |x(t)| \leqslant c \text{ and } |x(g(t))| \leqslant c \text{ for } t \geqslant T.$$

Hence

$$| f(t, x(t)), x(g(t)) | \leqslant p(t) H_1(c) + q(t) H_2(c).$$

It follows from the above inequality that

$$h(t) - [p(t) H_1(c) + q(t) H_2(c)] \leqslant h(t) - f(t, x(t), x(g(t))$$
$$\leqslant h(t) + [p(t) H_1(c) + q(t) H_2(c)]$$

for $t \geqslant T$. Thus

$$h(t) - [p(t) H_1(c) + q(t) H_2(c)] \leqslant L_n x(t) \leqslant h(t) + [p(t) H_1(c) + q(t) H_2(c)]. \quad ...(11)$$

Integrating (11) from T to t we obtain

$$\int_T^t [h(s) - (p(s) H_1(c) + q(s) H_2(c))] \, ds \leqslant L_{n-1} x(t)$$
$$\leqslant \int_T^t [h(s) + [p(s) H_1(c) + q(s) H_2(c))] \, ds.$$

Hence if either (9) or (10) holds, $x(t)$ cannot have arbitrarily large zero which is a contradiction and the proof of the theorem is complete.

Remark 4 : Almost all examples given in earlier papers[3,6,7,9] are linear but we have given several non-linear examples which are new to the literature.

3. Applications

The importance of the results obtained in the previous section will be shown by applying them to the particular case

$$r_i(t) = 1, \ i = 1, 2, ..., n.$$

In this case the equation (1) takes the form

$$x^{(n)}(t) + f(t, x(t), x(g(t))) = h(t), \ t > 0, \ n \geqslant 2. \quad ...(12)$$

Therefore, we have the following results.

Corollary 1—Suppose that (3) holds and that

$$\int^{\infty} [p(t) H_1(t^{n-1}) + q(t) H_2(t^{n-1})] \, dt < \infty \quad ...(13)$$

hold. Let $x(t)$ be a solution of (12). Then

$$x(g(t)) = O(t^{n-1}) \text{ as } t \to \infty.$$

Corollary 2—Let (13) and

$$\int^{\infty} t^{n-1} | h(t) | \, dt < \infty \quad ...(14)$$

hold. If $x(t)$ is an oscillatory solution of (8), then

$$\lim_{t \to \infty} x^{(k)}(t) = 0 \text{ for } k = 0, 1, \ldots, n-1.$$

Corollary 3—Suppose (3) and

$$\int^{\infty} t^{n-1} [p(t) H_1(t^{n-1}) + q(t) H_2(t^{n-1})] \, dt < \infty \qquad \ldots(15)$$

hold. Let

$$\lim_{t \to \infty} \frac{H(t)}{p(t) H_1(t^{n-1}) + q(t) H_2(t^{n-1})}$$

exist.

Then a necessary and sufficient condition for all oscillatory solutions of (12) to approach zero asymptotically is that

$$\lim_{t \to \infty} \frac{h(t)}{p(t) H_1(t^{n-1}) + q(t) H_2(t^{n-1})} = 0. \qquad \ldots(16)$$

Corollary 4—Let (3) and (15) hold. If

$$\frac{h(t)}{p(t) H_1(t^{n-1}) + q(t) H_2(t^{n-1})}$$

is bounded, then all oscillatory solutions of (12) tend to zero as $t \to \infty$.

Corollary 5—Let (3) and (15) hold. If

$$\lim_{t \to \infty} \inf \frac{h(t)}{p(t) H_1(t^{n-1}) + q(t) H_2(t^{n-1})} > 0$$

and

$$\frac{h(t)}{p(t) H_1(t^{n-1}) + q(t) H_2(t^{n-1})}$$

is bounded, then all solutions of (12) are non-oscillatory.

Remark : Theorems 7 and 10 of Grace and Lalli[4] are included in our Corollaries 1 and 3, respectively.

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ON BISHOP, ŠILOV AND ANTIALGEBRAIC DECOMPOSITIONS

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We discuss conditions under which the Bishop and Šilov decompositions for a function algebra A on a compact Hausdorff space X coincide. We discuss the antialgebraic and Šilov decompositions for a closed subspace M of $C_R(X)$ and also give analogous conditions for these two decompositions to be equal. Finally, we prove that the antialgebraic (Šilov) decomposition of the tensor product of two subspaces equals the product of the respective antialgebraic (Šilov) decompositions of the factor spaces.

1. INTRODUCTION

Let X be a compact Hausdorff space and $C(X)$ (resp. $C_R(X)$) denote the space of all continuous complex (resp. real) valued functions on X. A function algebra A on X is a closed subalgebra of $C(X)$ separating points of X and containing constants. The Bishop and Šilov decompositions of A are quite useful in characterizing A. Šilov[12] and Bishop[1] proved the basic property for the decompositions (which in our terminology is 'the (D)-property' [see Definition 2.3]). Since then Bishop decomposition has become an indespensable tool for proving many results for a function algebra. In most of the examples, these decompositions coincide; however there are examples where the Bishop decomposition is strictly finer than the Šilov decomposition (Gamelin[4], also see Sarason[10], p. 462). Very few results are known about the relation between these two decompositions. We discuss conditions under which these two decompositions coincide, in the second section of this paper.

Ellis[2] has defined and discussed the Bishop and Šilov decompositions for the subspace $A(K)$ of $C_R(K)$ of affine functions on a compact convex set K. Paltineanu[9] has defined an analogue of Bishop decomposition, called antialgebraic decomposition, for a closed subspace M of $C_R(X)$. In the third section, we define an analogue of Šilov decomposition for M and prove that this and the antialgebraic decompositions for M, play a role similar to the role of Šilov and Bishop decompositions for a function algebra A. Again, in general, the antialgebraic decomposition is finer than the Šilov decomposition and we prove results analogus to Section 2, giving conditions under which these decompositions coincide.

Tensor product of two algebraic structures is a tool to construct a similar new structure. The Bishop decomposition of the tensor product of two function algebras is known (Leibowitz[8], p. 221) to be the product of the Bishop decompositions of the factor algebras. In Section 4, we prove that similar results hold for the antialgebraic and Šilov decompositions of tensor product of closed subspaces.

2. Function Algebras

Throughout this Section A denotes a function algebra on X. A subset K of X is said to be a set of antisymmetry for A, if $f \in A$ and $f \mid_K$ is real valued implies $f \mid_K$ is constant. The collection of all maximal sets of antisymmetry forms a decomposition of X into closed sets (Leibowitz[8], p. 38) called the antisymmetric or Bishop decomposition of X with respect to A. It is denoted by $\mathscr{K}_A$.

Let $A_R = A \cap C_R(X)$. A maximal set of constancy for A_R is called a level set for A. The collection of all level sets for A also forms a decomposition of X, called the Šilov decomposition (Gamelin[4]) of X with respect to A. It is denoted by $\mathscr{F}_A$. If it is clear from the context which function algebra is discussed, we write $\mathscr{K}$ (resp. $\mathscr{F}$) in place of $\mathscr{K}_A$ (resp. $\mathscr{F}_A$). Before we prove the results, we need some terminology.

Definition 2.1—Let $\mathscr{E}_1$ and $\mathscr{E}_2$ be two families of subsets of X. If, for every $E_1 \in \mathscr{E}_1$, there exists $E_2 \in \mathscr{E}_2$ such that $E_1 \subset E_2$, then $\mathscr{E}_1$ is said to be finer than $\mathscr{E}_2$.

It is clear that if $\mathscr{E}_1$ and $\mathscr{E}_2$ are finer than each other and if members of each are pairwise disjoint, then $\mathscr{E}_1 = \mathscr{E}_2$.

Definition 2.2—A decomposition $\mathscr{E}$ of a topological space is said to be upper semicontinuous (u.s.c), if for each E in $\mathscr{E}$ and each open set U containing E there is an open set V such that $E \subset V \subset U$ and V is the union of members of $\mathscr{E}$.

Definition 2.3[6]—A family $\mathscr{E}$ of closed subsets of X is said to have the (D)-property for A, if $f \in C(X)$ and $f \mid_E \in (A \mid_E)^-$ for all $E \in \mathscr{E}$ implies $f \in A$: where $(A \mid_E)^-$ denotes the uniform closure of $A \mid_E$ in $C(E)$.

Remarks 2.4: (i) It is clear from the definition that $\mathscr{K}$ is finer than $\mathscr{F}$.

(ii) $\mathscr{F}$ is upper semicontinuous [Kaplan[7], p. 178].

(iii) $\mathscr{K}$ has the (D)-property for A by the Bishop's generalized Stone-Weierstrass theorem.

(iv) If $\mathscr{E}_1$ is finer then $\mathscr{E}_2$ and $\mathscr{E}_1$ has the (D)-property for A, then $\mathscr{E}_2$ also has the (D)-property for A. Consequently, $\mathscr{F}$ has the (D)-property for A.

As we have noted, in general $\mathscr{K}$ is finer than $\mathscr{F}$. But in most of the standard examples ($C(X)$, $A(D)$, antisymmetric algebras), $\mathscr{K} = \mathscr{F}$. However there are examples where these two decompositions are not equal[4]. We give certain conditions under which $\mathscr{K} = \mathscr{F}$.

Theorem 2.5—If $\mathscr{E}$ is an upper semicontinuous decomposition of X having the (D)-property for A, then $\mathscr{F}$ is finer than $\mathscr{E}$.

PROOF : Let $X/\mathscr{E}$ denote the quotient space of X obtained by $\mathscr{E}$ and $q : X \to X/\mathscr{E}$ the corresponding quotient map. Then, for $f \in C(X/\mathscr{E})$, $f \circ q \in C(X)$. Also $(f \circ q \mid _E$ is constant for each $E \in \mathscr{E}$. Therefore $(f \circ q) \mid _E \in A \mid _E$. Since $\mathscr{E}$ has the (D)- property for A, $f \circ q \in A$. Hence $f \circ q \in A_R$ for each $f \in C_R(X/\mathscr{E})$.

Let $F \in \mathscr{F}$ and $f \in C_R(X/\mathscr{E})$. Then $f \circ q$ is constant on F, i. e. f is constant on $q(F)$ for every $f \in C_R(X/\mathscr{E})$. Now $X/\mathscr{E}$ is Hausdorff as $\mathscr{E}$ is u.s.c. (Kaplan[7], p.177). Hence $q(F)$ must be a singleton set or $F \subset E$ for some $E \in \mathscr{E}$, which proves that $\mathscr{F}$ is finer than $\mathscr{E}$.

The following corollary is immediate.

Corollary 2.6 (Glicksberg[5], p. 433)—If $\mathscr{K}$ is u.s.c., then $\mathscr{K} = \mathscr{F}$.

It also follows from the above theorem that the Bishop decomposition determines the Šilov decomposition as the next corollary shows.

Corollary 2.7—Let A and B be function algebras on X. If $\mathscr{K}_A = \mathscr{K}_B$, then $\mathscr{F}_A = \mathscr{F}_B$.

PROOF : Since $\mathscr{K}_B$ is finer than $\mathscr{F}_A$, $\mathscr{F}_A$ has the (D)-Property for B. Also, $\mathscr{K}_A$ is upper semicontinuous. So by Theorem 2.5, $\mathscr{F}_B$ is finer than $\mathscr{F}_A$. By the same argument, $\mathscr{F}_A$ is finer than $\mathscr{F}_B$. Hence $\mathscr{F}_A = \mathscr{F}_B$.

The following example shows that the converse of the Corollary 2.7 is not true.

Example 2.2—Let $I = [0, 1]$ and $D = \{z \in \mathbb{C}; \mid z \mid \leqslant 1\}$. Let $X = \{r, z) \in I \times D; \mid z \mid \geqslant 1 - r/2\}$. For a fixed $r \in I$, let $X_r = \{(r, z); (r, z) \in X\}$. For $f \in C(X)$, let f_r be defined on X_r by $f_r(z) = f(r, z)$.

Let $A = \{f \in C(X); f_r$ is analytic on the interior of X_r, for $0 < r < 1\}$ and $B = \{f \in C(X); f_r$ can be uniformly approximated by polynomials on X_r, for $0 \leqslant r \leqslant 1\}$. Then A and B are function algebras on X. It can be checked that $\mathscr{F}_A = \{X_r; 0 \leqslant r \leqslant 1\} = \mathscr{F}_B = \mathscr{K}_B$. Further $\mathscr{K}_A = \{X_r; 0 < r \leqslant 1\} \cup \{\{(0, z)\}; \mid z \mid = 1\}$ [Glicksberg[5], Example 5.1]. Hence $\mathscr{F}_B = \mathscr{F}_A$ but $\mathscr{K}_B \neq \mathscr{K}_A$.

We give some additional results asserting the equality of $\mathscr{K}$ and $\mathscr{F}$.

Theorem 2.9—If $\mathscr{F}$ has finitely many members, then $\mathscr{K} = \mathscr{F}$.

PROOF : Suppose that $\mathscr{F} = \{F_1, F_2, \ldots, F_n\}$. We shall show that each F_i is a ret of antisymmetry, which proves the result

For F_1 and F_j, $j \neq 1$, there exists $f_{1j} \in A_R$ such that $f_{1j} = 1$ on F_1 and $f_{1j} = 0$ on F_j. Take $f_1 = f_{12}\ f_{13} \ldots f_{1n}$. Then $f_1 \in A_R$, $f_1 = 1$ on F_1 and $f_1 = 0$ on F_j for all $j \neq 1$. Similarly, for each $i = 2, 3, \ldots, n$, there exists $f_i \in A_R$ such that $f_i = 1$ on F_i and $f_i = 0$ on F_j for all $j \neq i$.

Let $g \in A$ and $g \mid_{F_i}$ be real valued. Then $h_i = gf_i$ is in A_R and hence h_i is constant on F_i. But $h_i = g$ on F_i and therefore g is constant on F_i. Thus F_i is a set of antisymmetry for A.

In view of Remark 2.4 (i) it is immediate from Theorem 2.9 that if $\mathscr{K}$ has finitely many members, then $\mathscr{K} = \mathscr{F}$. The following theorem shows that somewhat stronger result holds.

Theorem 2.10—If $\mathscr{K}$ has countably many members, then $\mathscr{K} = \mathscr{F}$.

PROOF : Suppose $\mathscr{K} = \{K_1, K_2, \ldots\}$. Let $F \in \mathscr{F}$. By Proposition 2 of Sidney[11], either $\mathscr{K}(A|_F) = \{F\}$ or $\mathscr{F}(A|_F)$ is uncountable. Now, since F is a level set, $\mathscr{K}(A|_F) = \mathscr{K} \cap F = \{K_i \cap F\,;\ i = 1, 2, \ldots\}$, i. e. $\mathscr{K}(A|_F)$ is almost countable; Therefore, $\mathscr{F}(A|_F)$ cannot be uncountable. Hence $\mathscr{F}(A|_F) = \{F\}$, which implies that $A \mid_F$ is an antisymmetric algebra. Thus $F \in \mathscr{K}$ or $\mathscr{K} = \mathscr{F}$.

Remark 2.11 : If $\mathscr{F}$ has countably many members then $\mathscr{K}$ and $\mathscr{F}$ may not be equal[4].

3. CLOSED SUBSPACES OF $C_R(X)$

We have discussed some results about the Bishop and Šilov decompositions for a function algebra on X. Analogous results hold for any closed subalgebra of $C(X)$ containing constants. Feyel[3] has defined the Bishop and Šilov decompositions for a subspace of $C(X)$. In case of a closed subspace of $C_R(X)$ containing constants, these decompositions coincide with the decompositions we consider below. Also, all the results which we prove remain true for a subspace of $C(X)$ without any essential change in the proof. Hence we restrict our discussion to subspaces of $C_R(X)$ only.

Let M be a closed subspace of $C_R(X)$ with $1 \in M$. For a subset S of X, define

$$G_M(S) = \{f \in M;\ \text{for each } g \in M,\ \exists\, h \in M \text{ such that } fg = h \text{ on } S\}.$$

Then it is easy to see that $G_M(S)$ is a closed subspace and $G_M(X)$ is an algebra. If M is fixed, we write $G(S)$ in place of $G_M(S)$.

Definition 3.1[9]—S is called an antialgebraic set for M, if $f \in G(S)$ implies f is constant on S.

The collection of all maximal antialgebraic sets for M is a decomposition of X (Paltineanu[9]) and is denoted by $\mathscr{S}_M$.

Definition 3.2—A maximal set of constancy for $G(X)$ is called a level set for M. We call the corresponding decomposition the Šilov decomposition for M and denote it by $\mathscr{C}_M$.

As in the second section we use $\mathscr{S}$ and $\mathscr{C}$ in place of $\mathscr{S}_M$ and $\mathscr{C}_M$.

Remarks 3.3 : (i) It is clear from the definition of $G(S)$ that if $S_1 \subset S_2$ then $G(S_2) \subset G(S_1)$. Consequently, $\mathscr{S}$ is finer than $\mathscr{C}$. If M is an algebra, then $\mathscr{S} = \mathscr{C}$ as $G(S) = M$ for each $S \subset X$.

(ii) $\mathscr{S}$ has the (D)-property w.r. to M (Paltineanu[9], Theorem 1). Consequently, $\mathscr{C}$, too, has the (D)-property for M.

(iii) $\mathscr{C}$ is an u.s.c. decomposition.

Thus the basic properties of $\mathscr{S}$ and $\mathscr{C}$ for M, correspond to the respective properties of $\mathscr{K}$ and $\mathscr{F}$ for a function algebra A. We will show that most of the results of section 2 hold true if $\mathscr{K}$ and $\mathscr{F}$ are replaced by $\mathscr{S}$ and $\mathscr{C}$.

Theorem 3.4—If ϵ is an u.s c. decomposition of X with (D)-property for M, then $\mathscr{C}$ is finer then $\mathscr{E}$.

PROOF : Let $F \in \mathscr{C}$. For $f \in C_R(X/\mathscr{E})$, we have $f \circ q \in C_R(X)$, where $q : X \to X/\mathscr{E}$ is the quotient map. Also $(f \circ q) \mid_E$ is constant, say α_E, for each $E \in \mathscr{E}$. Hence $f \circ q \in M$, by the (D)-property. To show that $f \circ q \in G(X)$, let $h \in M$. Then, $(f \circ q) h \mid_E = \alpha_F h \mid_F \in M \mid_F$ for all $E \in \mathscr{E}$. Since $\mathscr{E}$ has the (D)-property for M, $(f \circ q) h \in M$. Hence $f \circ q \in G(X)$ and so, as in the proof of Theorem 2.5, $F \subset E$ for some $E \in \mathscr{E}$.

Corollary 3.5—If $\mathscr{S}$ is u.s.c. then $\mathscr{S} = \mathscr{C}$.

Corollary 3.6—Let M_1 and M_2 be closed subspaces of $C_R(X)$ containing constants. If $\mathscr{S}_{M_1} = \mathscr{S}_{M_2}$, then $\mathscr{C}_{M_1} = \mathscr{C}_{M_2}$.

Note : We shall give an example [Example 3.9] to show that the converse of the Corollary 3.6 does not hold in general.

Theorem 3.7—If $\mathscr{C}$ consists of finitely many members, then $\mathscr{S} = \mathscr{C}$.

PROOF : Suppose that $\mathscr{C} = \{F_1, F_2, \ldots, F_n\}$. Now $G(X)$ is an algebra and hence as in the proof of Theorem 2.8, for each $i = 1, 2, \ldots, n$, there exists $f_i \in G(X)$ such that $f_i = 1$ on F_i and $f_i = 0$ on F_j for all $j \neq i$.

Let $g \in G(F_i)$. Since $f_i \in G(X)$, $gf_i \in M$. To show that $gf_i \in G(X)$, let $h \in M$, Then, $(gf_i) h \mid_{F_k} = (gh) \mid_{F_k} f_i \mid_{F_k} = 0$, if $k \neq i$ and $(gf_i) h \mid_{F_i} = gh \mid_{F_i} \in M \mid_{F_i}$, since $g \in G(F_i)$. So by the (D)-property of $\mathscr{C}$, $(gf_i)\, h \in M$. Hence $gf_i \in G(X)$. Therefore, gf_i is constant on F_i and hence g is constant of F_i. This proves that F_i is an antialgebraic set for M. Consequently, $\mathscr{S} = \mathscr{C}$.

Again in view of Remark 3.3 (i), it is clear from the above theorem that if $\mathscr{S}$ has finitely many members, then $\mathscr{S} = \mathscr{C}$. However, we do not know whether $\mathscr{S}$ and $\mathscr{C}$ concide if $\mathscr{S}$ has countably many members.

To give the promised example showing the falsehood of the converse of the Corollary 3.6, we need to develop certain machinery.

As we have mentioned in the introduction, the Bishop $\mathscr{K}_{A(K)}$ and Šilov ($\mathscr{F}_{A(K)}$) decompositions for $A(K)$ have been discussed by Ellis[2]. In fact, these are decompositions of ∂K, the set of extreme points of K. The members of $\mathscr{K}_{A(K)}$ and $\mathscr{F}_{A(K)}$, are closed split faces of K.

Now, let A be a function algebra on X and S the State space (Ellis[2], p. 564) of A, $Z = \overline{co}\ (S \cup (-iS))$, the closed convex hull of $S \cup (-iS)$. Then there exists a topological isomorphism of A onto $A(Z)$ and each $K \in \mathscr{K}_A$ ($F \in \mathscr{F}_A$) is of the form $K = K' \cap X$ ($F = F' \cap X$) for some K' in $\mathscr{K}_{A(Z)}$ ($F' \in \mathscr{F}_{Z(Z)}$). It can be seen further that this correspondence between $\mathscr{K}_A$ and $\mathscr{K}_{A(Z)}$ (resp. $\mathscr{F}_A$ and $\mathscr{F}_{A(Z)}$) is one-one also, for if F_1 and F_2 are closed faces of Z such that $F_1 \cap X = F_2 \cap X$ then $F_1 = F_2$. Consequently, we get

Proposition 3.8—$\mathscr{K}_A = \mathscr{F}_A$ iff $\mathscr{K}_{A(Z)} = \mathscr{F}_{A(Z)}$.

Ellis[2] has also defined a decomposition of 'Maximal weak sets of antisymmetry' for $A(K)$. In case of $A(Z)$, it coincides with $\mathscr{K}_{A(Z)}$. Thus the family of maximal weak sets of antisymmetry for $A(Z)$ in $\overline{\partial Z}$ is the family $\{F \cap \overline{\partial Z};\ F \in \mathscr{K}_{A(Z)}\}$ $= \mathscr{K}_{A(Z)} \cap \overline{\partial Z}$. It is shown in [Paltineanu[9], Remark 4] that if $M = A(K)\ \overline{\partial K}$ then $\mathscr{S}_M$ coincides with the family of maximal weak sets of antisymmetry for $A(K)$ in $\overline{\partial K}$. Therefore, for $M = A(Z) \mid_{\overline{\partial Z}}$, $\mathscr{S}_M = \mathscr{K}_{A(Z)} \cap \overline{\partial Z}$. Also the family $\mathscr{F}_{A(Z)} \cap \overline{\partial Z}$ forms the maximal sets of constancy of the central functions in $A(Z)$. Since $G(\partial Z)$ = the restriction of central functions in $A(Z)$ to $\overline{\partial Z}$, $\mathscr{C}_M = \mathscr{F}_{A(Z)} \cap \overline{\partial Z}$.

Now we are ready to give the example (Gamelin[4]).

Example 3.9—Let A be the function algebra (Gamelin[4]) for which $\mathscr{K}_A \neq \mathscr{F}_A$. Then, by Proposition 3.8, $\mathscr{K}_{A(Z)} \neq \mathscr{F}_A$. Since the decompositions consits of closed faces of Z, it can be easily seen that, $\mathscr{K}_{A(Z)} \cap \overline{\partial Z} \neq \mathscr{F}_{A(Z)} \cap \overline{\partial Z}$. Now, let $M_1 = A(Z \mid_{\overline{\partial Z}}$ and $M_2 = G_{M_1}(\overline{\partial Z})$, Then $\mathscr{S}_{M_1} = \mathscr{K}_{A(Z)} \cap \overline{\partial Z} \neq \mathscr{F}_{A(Z)} \cap \overline{\partial Z} = \mathscr{C}_{M_1}$. Also $\mathscr{S}_{M_2} = \mathscr{C}_{M_2}$. But $M_2 = G_{M_2}(\overline{\partial Z})$, and hence $\mathscr{C}_{M_2} = \mathscr{C}_{M_1}$. Thus $\mathscr{C}_{M_1} = \mathscr{C}_{M_2}$ but $\mathscr{S}_{M_1} \neq \mathscr{S}_{M_2}$

4. Tensor Product

The tensor product of two function algebras is a function algebra and its properties have been discussed in Leibowitz[8] (p. 218). In particular it is known that $\mathscr{K}_{A \hat{\otimes} B} = \mathscr{K}_B \times \mathscr{K}_B$ (Leibowitz[8], p. 221). On the some line, it can be shown that $\mathscr{F}_{A \hat{\otimes} B} = \mathscr{F}_A \times \mathscr{F}_B$.

Now the tensor product can also be defined for subspaces. So, the natural question is 'what about the corresponding antialgebraic and Šilov decompositions ?' We prove that $\mathscr{S}_{M \hat{\otimes} N} = \mathscr{S}_M \times \mathscr{S}_N$ and $\mathscr{C}_{M \hat{\otimes} N} = \mathscr{C}_M \times \mathscr{C}_N$, where M and N are closed subspaces of $C_R(X)$ and $C_R(Y)$ respectively, containing constants. As usual, $M \hat{\otimes} N$ denotes the uniform closure of the space of all finite linear combinations of functions $\{f \otimes g;\ f \in M,\ g \in N\}$, where $f \otimes g$ is a function on $X \times Y$ defined by $(f \otimes g)(x, y) = f(x)\, g(y)$. It is easy to see that $M \hat{\otimes} N$ contains constants. For $f \in M \hat{\otimes} N$, $f_x \in N \ \forall\ x \in X$ and $f_y \in M \ \forall\ y \in Y$, where $f_x(y) = f(x, y)\ (y \in Y)$ and $f_y(x) = f(x, y)\ (x \in X)$.

Lemma 4.1—Let $S \subset X$ and $T \subset Y$.

(i) If $f \in G_{M \hat{\otimes} N}(S \times T)$ then $f_x \in G_N(T)$ and

$f_y \in G_M(S)$ for each $x \in S$ and $y \in T$.

(ii) $G_M(S) \hat{\otimes} G_N(T) \subset G_{M \hat{\otimes} N}(S \times T)$.

Proof : Since there is no likelihood of confusion, we write $G(S \times T)$, $G(S)$ and $G(T)$ in place of $G_{M \hat{\otimes} N}(S \times T)$, $G_M(S)$ and $G_N(T)$.

(i) Clearly $f_x \in N$. Let $g \in N$. Then $1 \otimes g \in M \hat{\otimes} N$. Since $f \in G(S \times T)$, there exists $h \in M \hat{\otimes} N$ such that $f(1 \otimes g) = h$ on $S \times T$. Then $h_x \in N$ and $(f(1 \otimes g))_x = h_x$ on T for $x \in S$. But $(f(1 \otimes g))_x = f_x\, g$. Therefore, $f_x\, g = h_x$ on T for each $x \in S$. Hence $f_x \in G(T)$ for each $x \in S$. By the same argument one can show that $f_y \in G(S)$ for $y \in T$.

(ii) Let $\phi \in G(S) \otimes G(T)$. Then $\phi = \sum_i f_i \otimes g_i$, where $f_i \in G(S)$ and $g_i \in G(T)$.

Clearly, $\phi \in M \hat{\otimes} N$. Let $\psi \in M \otimes N$. Then $\psi = \sum_j p_j \otimes q_j$, where $p_j \in M$ and $q_j \in N$. Now $f_i \in G(S)$ and $p_j \in M$. So, there exists $h_{ij} \in M$ such that $h_{ij} = f_i\, p_j$ on S. Similarly, there exists $h'_{ij} \in N$ such that $g_j\, q_j = h'_{ij}$ on T. Take $h = \sum_{i,j} h_{ij} \otimes h'_{ij}$. Then $h \in M \otimes N$ and $\phi\, \psi = h$ on $S \times T$. Therefore, since $M \otimes N$ is dense in $M \hat{\otimes} N$, it follows that $\phi \in G(S \times T)$, i. e., $G(S) \otimes G(T) \subset G(S \times T)$. Hence, $G(S) \hat{\otimes} G(T) \subset G(S \times T)$.

Finally, we prove the results regarding decompositions for the tensor product.

Theorem 4.2—Let M and N be closed subspaces of $C_R(X)$ and $C_R(Y)$ respectively, containing constants. Then

(i) $\mathscr{S}_{M \hat{\otimes} N} = \mathscr{S}_M \times \mathscr{S}_N$

and

(ii) $\mathscr{C}_{M \hat{\otimes} N} = \mathscr{C}_M \times \mathscr{C}_N$.

PROOF : (i) Let $S \in \mathscr{S}_M$, $T \in \mathscr{S}_N$ and $f \in G(S \times T)$. Then, by the Lemma 4.1 (i), $f_x \in G(T)$ and $f_y \in G(S)$ for $x \in S$ and $y \in T$. Therefore, f_x is constant on T and f_y is constant on S, for $x \in S$, $y \in T$. Hence f is constant on $S \times T$, i. e. $S \times T$ is an antialgebraic set for $M \hat{\otimes} N$. Hence, there exists $H \in \mathscr{S}_{M \hat{\otimes} N}$ such that $S \times T \subset H$.

Let p_x and p_y denote the projections of $X \times Y$ onto X and Y, respectively. Let $g \in G(p_x(H))$. Then, by the Lemma 4.1 (ii), $g \otimes \in G(p_x(H) \times p_y(H))$. So, $g \otimes 1 \in G(H)$. Since $H \in \mathscr{S}_{M \hat{\otimes} N}$, $g \otimes 1$ is constant on H and hence g is constant on $p_x(H)$, i. e. $p_x(H)$ is an antialgebraic set for M. But $S \subset p_x(H)$ and $S \in \mathscr{S}_M$. Therefore $S = p_x(H)$. Similarly, $T = p_y(H)$. Thus $S \times T = H$.

(ii) By the Lemma 4.1 (ii), we have $G(X) \hat{\otimes} G(Y) \subset G(X \times Y)$. Hence $\mathscr{C}_{M \hat{\otimes} N}$ is finer than $\mathscr{C}_M \times \mathscr{C}_N$.

Conversely, suppose $F \in \mathscr{C}_M$ and $K \in \mathscr{C}_N$. Let $f \in G(X \times Y)$. Then, by the Lemma 4.1 (i), $f_x \in G(Y)$ and $f_y \in G(X)$ for $x \in X$, $y \in Y$. So, f_x is constant on F and f_y is constant on K for $x \in X$, $y \in Y$. Hence f is constant on $F \times K$, i. e. $F \times K$ is a set of constancy for $G(X \times Y)$. Therefore, $\mathscr{C}_M \times \mathscr{C}_N$ is finer then $\mathscr{C}_{M \hat{\otimes} N}$. Consequently, $\mathscr{C}_M \times \mathscr{C}_N = \mathscr{C}_{M \hat{\otimes} N}$.

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ON TOPOLOGICAL PROJECTIVE PLANES-III

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Salzmann[4] has proved the following theorem: In a topological projective plane the connectedness of

(a) the space of points

(b) the space of lines

(c) a ray of points

(d) a ray of affine points

(e) the set of points of an affine plane

are all equivalent. In this paper we prove a similar theorem for several other topological properties.

1. INTRODUCTION

A topological projective plane is a projective plane in which the set of points $\mathscr{P}$ and the set of lines $\mathscr{L}$ are endowed with topologies $\tau_{\mathscr{P}}$ and $\tau_{\mathscr{L}}$ respectively such that the operations of joining and intersecting are continuous in both variables. If either $\tau_{\mathscr{P}}$ or $\tau_{\mathscr{L}}$ is different from the indiscrete topology, then both $\tau_{\mathscr{P}}$ and $\tau_{\mathscr{L}}$ are Hausdorff. Hence we assume throughout that all the spaces involved are Hausdorff. Salzmann[4] has proved the following theorem.

In a topological projective plane the connectedness of

(a) the space of points

(b) the space of lines

(c) a ray of points

(d) a ray of affine points

(e) the set of points of an affine plane

are all equivalent.

In this paper we prove a similar theorem for several other topological properties. For topological terminology we refer Dugundji[2]. The following theorems are used in the sequel.

Theorem 1.1[3]—$(\mathscr{P}, \tau_{\mathscr{P}})$ and $(\mathscr{L}, \tau_{\mathscr{L}})$ are always regular topological spaces.

Theorem 1.2[3]—In a topological projective plane each ray of points and each pencil of lines are closed.

Theorem 1.3[3]—In a topological projective plane the relative topology on a ray of points and a pencil of lines are homeomorphic.

Theorem 1.4[3]—In a topological projective plane the set of points of an affine plane is homeomorphic to the product of an affine ray with itself.

2. Main Results

Theorem 2.1—In a topological projective plane the path connectedness of

(a) the space of points

(b) the space of lines

(c) a ray of points

(d) a ray of affine points

(e) the set of points of an affine plane

are all equivalent.

Proof : The product of two topological spaces is path connected iff each is path connected. Hence by Theorem 1.4, the path connectedness of (d) and (e) are equivalent. Also union of any family of path connected spaces having nonempty intersection is path connected and hence the path connectedness of (d) implies that of (e) and (a) and the path connectedness of (c) implies that of (a). Now, to prove that the path connectedness of (a) implies that of (c), suppose that the ray of points of a line l is not path connected. Let $P \in l$. The path component of P in l is left unchanged by all homeomorphisms of the ray of points of l which fix P and through such homeomorphisms any point $\neq P$ on l can be mapped to any point $\neq P$ on l. Hence the path component of P in l is $\{P\}$. Now, let Q be any point not on l and let $\mathscr{D}$ be the path component of P in $\mathscr{P}$—$\{Q\}$. Suppose $\mathscr{D}$ contains a point R not lying on the line PQ. Since $X \to Q\, X \cap l$ is continuous at all points $\neq Q$ and the continuous image of a path connected sapce is path connected, the image of $\mathscr{D}$ under this map is a path connected subset of l containing P and $QR \cap l$. Thus the path component of P in l contains more than one point which is a contradiction. Hence $\mathscr{D} \subseteq PQ$—$\{Q\}$. But the path component of P in any ray of points is $\{P\}$ and hence $\mathscr{D} = \{P\}$. Thus for any arbitrary point $Q \neq P$, the path component of P in $\mathscr{P} - \{Q\}$ is $\{P\}$ and hence P cannot be joined to any other point by a path in $\mathscr{P}$. Hence $\mathscr{P}$ is not path connected and thus the path connectedness of (a) implies that of (c). Similarly we can prove that path connectedness of (c) implies that of (d). Now, suppose that the space of points is path connected. Then a ray of points is path connected and hence by Theorem 1.2, each pencil of lines is path connected. Hence the space of lines is path

connected. Thus the path connectedness of (a) implies that of (b). Similarly the path connectedness of (b) implies that of (a).

Corollary 2.2—A topological projective plane which is not path connected is totally path disconnected.

Theorem 2.3—In a topological projective plane, the local connectedness, the property of having a σ-locally finite base, second countability, separability and first countability of

(a) the space of points

(b) the space of lines

(c) a ray of points

(d) a ray of affine points

(e) the set of points of an affine plane

are all equivalent.

PROOF : We prove the theorem for locally connectedness and the proof for the remaining properties is similar. Since an open subspace of a locally connected space is locally connected, the local connectedness of (a) implies that of (e) by Theorem 1.2. Now suppose that the set of points of any affine plane is locally connected. Let OEUV be the quadrangle of reference. If $\mathscr{B}_1$, $\mathscr{B}_2$, $\mathscr{B}_3$ are bases consisting of open connected sets for the affine planes $\mathscr{P} - VU$, $\mathscr{P} - OV$ and $\mathscr{P} - OU$ respectively, then $\mathscr{B}_1 \cup \mathscr{B}_2 \cup \mathscr{B}_3$ is a base consisting of open connected sets for $\mathscr{P}$. Thus the local connectedness of (a) and (e) are equivalent. By a similar argument we can prove the equivalence of the local connectedness of (c) and (d). We now prove the equivalence of the local connectedness of (a) and (b). Suppose that the space of lines $\mathscr{L}$ is locally connected. By Theorem 1.2 the set $\mathscr{L}_1$ of all lines not passing through a point V is an open set in $\mathscr{L}$ and hence is locally connected. Also $\mathscr{L}_1$ is homeomorphic to the set of points of an affine plane. Hence the set of points of an affine plane is locally connected. Hence $\mathscr{P}$ is locally connected. Thus the local connectedness of (b) implies that of (a). Similarly if $\mathscr{P}$ is locally connected, if follows that the complement of any pencil of lines in $\mathscr{L}$ is locally connected and hence $\mathscr{L}$ is locally connected. The equivalence of the local connectedness of (d) and (e) follows from Theorem 1.4.

Theorem 2.4—In a topological projective plane the metrizability of

(a) the space of points

(b) the space of lines

(c) a ray of points

(d) a ray of affine points

(e) the set of points of an affine plane

are all equivalent.

PROOF : A topological space is metrizable iff it is regular and has σ-locally finite base (Dugundji[2], page 194). Hence the result follows from Theorem 2.3.

3. CONCLUSION

No known topological projective planes have non-homeomorphic topologies on the space of points and lines. In Theorems 2.1, 2.3 and 2.4 we have proved for several topological properties P that the space of points has property P iff the space of lines has P. Further in[1] we have proved that in a topological pappian plane the space of points is homeomorphic to the space of lines. Wyler[5] has proved a similar theorem for ordered projective planes. Hence we take the risk of making the following.

Conjecture—In a topological projective plane the space of points is homeomorphic to the space of lines.

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INVARIANT SUBMANIFOLDS IN A CONFORMAL K-CONTACT RIEMANNIAN MANIFOLD

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Kon[1] gave necessary conditions for an invariant submanifold M of a K-contact Riemannian manifold $\bar{M}$ to be minimal and gave the necessary and sufficient conditions for M to be totally geodesic. Further Endo[2] also has shown that an invariant submanifold M of a K-contact Riemannian manifold $\bar{M}$ is minimal. Here we give some conditions for the invariant submanifold of a conformal K-contact Riemannian manifold to be minimal and totally geodesic.

1. PRELIMINARIES

A conformal K-contact Riemannian manifold $\bar{M}$ is defined[3] as an almost Sasakian manifold with structure tensors $\bar{\phi}$, $\bar{\xi}$, $\bar{\eta}$, $\bar{g}$ in which the associated vector field $\bar{\xi}$ is a conformal killing vector field i. e. in which

$$(\bar{\nabla}_{\bar{X}}\bar{\eta})\ (\bar{Y}) + (\bar{\nabla}_{\bar{Y}}\bar{\eta})\ (\bar{X}) = 2\alpha\ \bar{g}\ (\bar{X}, \bar{Y})$$

where $\bar{\nabla}$ is the Riemannian connection, $\bar{X}$, $\bar{Y}$ are vector fields on $\bar{M}$ and α is a scalar. If, in particular, $\alpha = 0$, then the manifold is a K-contact manifold[4].

For a conformal K-contact Riemannian manifold $\bar{M}$, we have[3]

$$\bar{\phi}\ \bar{\xi} = 0,\ \bar{\eta}\ (\bar{\xi}) = 1,\ \bar{\phi}^2\ \bar{X} = -\bar{X} + \bar{\eta}\,(\bar{X})\ \bar{\xi} \qquad \ldots(1.1)$$

$$\bar{g}\ (\bar{\phi}\ \bar{X},\ \bar{\phi}\ \bar{Y}) = \bar{g}\ (\bar{X}, \bar{Y}) - \bar{\eta}\,(\bar{X})\,\bar{\eta}\ (\bar{Y}),\ \bar{\eta}\,(\bar{X}) = \bar{g}\ (\bar{\xi},\ \bar{X}) \qquad \ldots(1.2)$$

for any vector fields $\bar{X}$ and $\bar{Y}$ on $\bar{M}$.

It is known[3] that in a conformal K-contact Riemannian manifold, the following formulas hold :

$$\bar{\Phi}\,(\bar{X}, \bar{Y}) = \alpha\ \bar{g}\ (\bar{X}, \bar{Y}) - (\nabla_{\bar{Y}}\bar{\eta})\,(\bar{X}) = (\bar{\nabla}_{\bar{X}}\bar{\eta})\,(\bar{Y}) - \alpha\ \bar{g}\ (\bar{X}, \bar{Y}) \qquad \ldots(1.3)$$

$$\bar{K}(\bar{X}, \bar{Y}, \bar{Z}, \bar{\xi}) + \bar{K}(\bar{\phi}\bar{X}, \bar{\phi}\bar{Y}, \bar{Z}, \bar{\xi}) = \bar{\Phi}(\bar{X}, \bar{Z})(\bar{\phi}\bar{Y}\alpha) - \bar{\Phi}(\bar{Y}, \bar{Z})(\bar{\phi}\bar{X}\alpha) + \bar{g}(\bar{X}, \bar{Z})(\bar{Y}\alpha) - \bar{g}(\bar{Y}, \bar{Z})(\bar{X}\alpha) + \bar{\eta}(\bar{Y})\bar{\eta}(\bar{\nabla}_{\bar{Z}}\bar{\phi}\bar{X}) - \bar{\eta}(\bar{X})\bar{\eta}(\bar{\nabla}_{\bar{Z}}\bar{\phi}\bar{Y}) \quad ...(1.4)$$

$$\bar{g}(\bar{X}, \bar{Y}) + \alpha\bar{\Phi}(\bar{X}, \bar{Y}) + \bar{\eta}(\bar{\nabla}_{\bar{Y}}\bar{\phi}\bar{X}) = \bar{\eta}(\bar{X})\bar{\eta}(\bar{Y}) \quad ...(1.5)$$

where $\bar{K}$ is the Riemannian curvature tensor on $\bar{M}$ and $\bar{\Phi}(\bar{X}, \bar{Y}) = \bar{g}(\bar{\phi}\bar{X}, \bar{Y})$.

By virtue of the above relations, we have

$$\bar{\nabla}_{\bar{X}}\bar{\xi} = \bar{\phi}\bar{X} + \alpha\bar{X} \quad ...(1.6)$$

$$\bar{K}(\bar{X}, \bar{\xi})\bar{Y} = (\bar{\nabla}_{\bar{X}}\bar{\phi})\bar{Y} + (\bar{\xi}\alpha)\bar{g}(\bar{X}, Y)\bar{\xi} - (\bar{X}\alpha)\bar{Y} \quad ...(1.7)$$

$$\bar{K}(\bar{X}, \bar{\xi})\bar{\xi} = \bar{X} - \bar{\eta}(\bar{X})\bar{\xi} + (\bar{\xi}\alpha)\bar{\eta}(\bar{X})\bar{\xi} - (\bar{X}\alpha)\bar{\xi}. \quad ...(1.8)$$

Let M be an $(2m+1)$ dimensional $(n > m)$ manifold imbedded in $\bar{M}$. The induced metric g of M is given by $g(X, Y) = \bar{g}(\bar{X}, \bar{Y})$ for any vector fields X, Y on M. Let $T_x(M)$ and $T_x(M)^{\perp}$ denote that tangent and normal bundles of M and $x \in M$. Let ∇_X denote the Riemannian connection on M determined by the induced metric g and K denote the Riemannian curvature tensor of M. Then Gauss-Weingarten formula is given by

$$\bar{\nabla}_X Y = \nabla_X Y + B(X, Y), \quad \bar{\nabla}_X N = -A_N(X) + D_X N \quad ...(1.9)$$

for any vector fields X, Y tangent to M and any vector field N normal to M, where D is the operator of covariant differentiation with respect to the linear connection induced in the normal bundle $T_X(M)^{\perp}$. Both A and B are called the second fundamental forms of M and they satisfy $\bar{g}(B(X, Y), N) = g(A_N(X, (Y)$.

A submanifold M of $\bar{M}$ is said to be invariant if $\bar{\xi}$ tangent to M everywhere on M and $\bar{\phi}X$ is tangent to M for any tangent vector X to M. An invariant submanifold M has the induced structure tensors (ϕ, ξ, η, g).

2. Invariant Submanifolds in Conformal K-contact Riemannian Manifold

Let $\bar{M}$ be a $(2n+1)$ dimensional conformal K-contact Riemannian manifold and M a $(2m+1)$ dimensional $(n > m)$ manifold imbedded in $\bar{M}$.

For the second fundamental form B of an invariant submanifold M of a conformal K-contact Riemannian manifold, we define its covariant derivative $(\tilde{\nabla}_X B)$ by

$$(\tilde{\nabla}_X B)(Y, Z) = D_X (B(Y, Z)) - B(\nabla_X Y, Z) - B(Y, \nabla_X Z). \qquad \text{...(2.1)}$$

$X, Y, Z \in X(M)$— the set of all differentiable vector fields on M.

Then by (1.9), we obtain

$$\bar{K}(X, Y) Z = K(X, Y) Z - A_{B(Y,Z)}(X) + A_{B(X,Z)}(Y)$$

$$+ (\tilde{\nabla}_X B)(Y, Z) - (\tilde{\nabla}_Y B)(X, Z).$$

Lemma 2.1—If M is an invariant submanifold of a conforml K-contact Riemannian manifold $\bar{M}$, then its second fundamental form B satisfies $B(X, \xi) = 0$ for any $X \in X(M)$.

PROOF : Since $\bar{\xi}$ is tangent to M everywhere on M, we have

$$\bar{\nabla}_X \bar{\xi} = \bar{\nabla}_X \xi = \nabla_X \xi + B(X, \xi). \qquad \text{...(2.3)}$$

Since by (1.6), $\bar{\Delta}_X \xi$ is tangent to M for any $X \in X(M)$, then by taking the normal parts of 2.3), we get $B(X, \xi) = 0$.

Lemma 2.2 —Any invariant submanifold M with induced structure tensors of a conformal K-contact Riemannian manifold $\bar{M}$ is also conformal K-contact.

PROOF : From (2.2) and Lemma 2.1, we have

$$\bar{K}(X, \xi) \xi = K(X, \xi) \xi + (\tilde{\nabla}_X B)(\xi, \xi) - (\tilde{\nabla}_\xi B)(X, \xi). \qquad \text{...(2.4)}$$

Again from (2.1) and Lemma 2.1, we get

$$(\tilde{\nabla}_X B)(\xi, \xi) = 0 \qquad \text{...(2.5)}$$

and

$$(\tilde{\nabla}_\xi B)(X, \xi) = 0. \qquad \text{...(26)}$$

Finally using (2.5) (2.6) and (1.7) in (2.4) we obtain

$$K(X, \xi) \xi = \bar{K}(X, \xi) \xi = X - \eta(X) \xi + (\xi \alpha) \eta(X) \xi - (X\alpha) \xi.$$

This shows that M is a conformal K-contact Riemannian manifold.

Lemma 2.3—Let M be an invariant submanifold of a conformal K-contact Riemannian manifold $\bar{M}$. Then $\bar{K}(X, \xi) Y$ is tangent to M if and only if $\bar{\phi} B(X, Y) = B(X, \phi Y)$ for any $(X, Y \in X(M)$.

PROOF : It is seen in Lemma 2.2 that M is also conformal K-contact Riemannian manifold. Since ξ is a conformal killing vector field on M and $\bar{M}$, by virtue of (1.7) we have

$$\bar{K}(X, \bar{\xi})Y = (\bar{\nabla}_X \bar{\phi})Y + \bar{g}(X, Y)(\bar{\xi}\alpha)\bar{\xi} - (X\alpha)Y \quad \text{...(2.7)}$$

and

$$K(X, \xi)Y = (\nabla_X \phi)Y + g(X, Y)(\xi\alpha)\xi - (X\alpha)Y. \quad \text{...(2.8)}$$

On the otherhand, from (1.9) we have

$$\bar{\nabla}_X(\bar{\phi}Y) = \nabla_X(\phi Y) + B(X, \phi Y) = (\nabla_X \phi)Y + \phi(\nabla_X Y) + B(X, \phi Y) \quad \text{...(2.9)}$$

and

$$\bar{\nabla}_X(\bar{\phi}Y) = (\bar{\nabla}_X \bar{\phi})Y + \bar{\phi}(\bar{\nabla}_X Y) = (\bar{\nabla}_X \bar{\phi})Y + \bar{\phi}(\nabla_X Y + B(X, Y))$$

$$= (\bar{\nabla}_X \bar{\phi})Y + \phi(\nabla_X Y) + \bar{\phi}B(X, Y). \quad \text{...(2.10)}$$

From (2.7), (2.8), (2.9) and (2,10) we have

$$K(X, \xi)Y - g(X, Y)(\xi\alpha)\xi - (X\alpha)Y - \bar{K}(X, \xi)Y$$

$$+ \bar{g}(X, Y)(\bar{\xi}\alpha)\bar{\xi} + (X\alpha)Y = \bar{\phi}B(X, Y) - B(X, \phi Y)$$

which implies

$$K(X, \xi)Y - \bar{K}(X, \xi)Y = \bar{\phi}B(X, Y) - B(X, \phi Y)$$

Hence the Lemma. Q.E.D.

Lemma 2.4—For an invariant submanifold M of a conformal K-contact Riemannian manifold $\bar{M}$, if the vector field X on M is orthogonal to $\bar{\xi}$, then we have

$$\bar{\phi}\bar{K}(\bar{\xi}, N)X = -\bar{K}(\xi, N)\bar{\phi}X - 2(N\alpha)\bar{\phi}X$$

where N is a vector field normal to M.

Proof : From (1.1), we find

$$\bar{\nabla}_N(\bar{\phi}^2 X) = -\bar{\nabla}_N X + \bar{g}(\bar{\nabla}_N X, \bar{\xi})\bar{\xi} + \bar{g}(X, \bar{\nabla}_N \bar{\xi})\bar{\xi}$$

$$+ \bar{g}(X, \bar{\xi})\bar{\nabla}_N \bar{\xi}.$$

On the other hand, we have

$$\bar{\nabla}_N(\bar{\phi}^2 X) = (\bar{\nabla}_N \bar{\phi})\bar{\phi}X + \bar{\phi}(\bar{\nabla}_N \bar{\phi})X + \bar{\phi}^2\bar{\nabla}_N X$$

$$= \bar{K}(N, \bar{\xi})\bar{\phi}X - (\bar{\xi}\alpha)g(N, \bar{\phi}X) + (N\alpha)\bar{\phi}X$$

$$+ \bar{\phi}\bar{K}(N, \bar{\xi})X - (\bar{\xi}\alpha)\bar{g}(N, X)\bar{\phi}\bar{\xi} + (N\alpha)\bar{\phi}X$$

$$- \bar{\nabla}_N X + \bar{g}(\bar{\nabla}_N X, \bar{\xi})\bar{\xi}.$$

Thus we have,

$$\bar{g}(X, \bar{\nabla}_N \bar{\xi})\, \bar{\xi} + \bar{g}(X, \bar{\xi})\, \bar{\nabla}_N \bar{\xi} = \bar{K}(N, \bar{\xi})\, \bar{\phi} X - (\bar{\xi}\alpha)\, \bar{g}(N, \bar{\phi} X)\, \bar{\xi}$$
$$+ 2(N\alpha)\, \bar{\phi} X + \bar{\phi}\, \bar{K}(N, \bar{\xi})\, X$$

from which

$$0 = \bar{\eta}(X)\,(\bar{\phi} N + \alpha N) + \bar{g}(X, \bar{\phi} N + \alpha N)\, \bar{\xi}$$
$$= \bar{K}(N, \bar{\xi})\, \bar{\phi} X + \bar{\phi}\, \bar{K}(N, \bar{\xi})\, X + (\bar{\xi}\alpha)\, \bar{g}(N, \bar{\phi} X)\, \bar{\xi} + 2(N\alpha)\, \bar{\phi} X$$
$$= \bar{K}(N, \bar{\xi})\, \bar{\phi} X + \bar{\phi}\, \bar{K}(N, \bar{\xi})\, X + 2(N\alpha)\, \bar{\phi} X$$

Theorem 2.1—Any invariant submanifold M of a conformal K-contact Riemannian manifold $\bar{M}$ is minimal.

PROOF : From the proof of Lemma 2.3, we have

$$\bar{g}(\bar{K}(X, \xi)\, Y, N) = \bar{g}(B(X, \phi Y), N) - \bar{g}(\bar{\phi}\, B(X, Y), N)$$
$$= \bar{g}(A_N X, \phi Y) + \bar{g}(A_{\bar{\phi} N} X, Y)$$

where A_N is defined to be $g(B(X, Y), N) = g(A_N X, Y)$.

Replacing Y by ϕY, we find

$$\bar{g}(A_N X, Y) = -\, \bar{g}(\bar{K}(X, \xi)\, \phi Y, N) + g(A_{\bar{\phi} N} X, \phi Y).$$

Taking a ϕ-basis $\{\xi;\, e_1, \ldots, e_m : \phi e_1, \ldots, \phi e_m\}$

we have

$$\mathrm{Tr}\, A_N = -\sum_{i=1}^{m} \bar{g}(\bar{K}(e_i, \xi)\, \phi e_i, N)$$
$$- \sum_{i=1}^{m} \bar{g}\, \bar{K}(\phi e_i, \xi)\, \phi^2 e_i, N) - \mathrm{Tr}\, A_N \phi$$
$$= \sum_{i=1}^{m} \bar{g}(\bar{K}(\phi e_i, e_i)\, \xi, N)$$

since ϕ is skew-symmetric, $\mathrm{Tr}\, A_N \phi$ vanishes identically and by virtue of Bianchi's identity.

On the other hand, by Lemma 2.4, we get

$$\bar{g}(\bar{K}(\phi e_i, e_i)\, \xi, N) = \bar{g}(\bar{K}(\xi, N)\, \phi e_i, e_i)$$
$$= -\, \bar{g}(\bar{\phi}\, K(\xi, N)\, e_i, e_i) - 2(N\alpha)\, g(\phi e_i, e_i)$$

(*equation continued on p.* 1124)

$$= -\ \bar{g}\,(\bar{\phi} K(\xi,\ N)\, e_i,\ e_i)$$

$$= \bar{g}\ (\bar{K}(\xi,\ N)\, e_i,\ \phi\, e_i)$$

$$= \bar{g}\,(\bar{K}(e_i,\ \phi\, e_i),\ \xi,\ N)$$

$$= -\ \bar{g}\ (\bar{K}(e_i,\ e_i)\ \xi,\ N)$$

from which $\bar{g}\,(\bar{K}(\phi\, e_i,\ e_i)\ \xi,\ N) = 0$. Thus we have $\operatorname{Tr} A_N = 0$ for all N, i. e. $\operatorname{Tr} B = 0$, which shows that M is minimal

We consider the second fundamental form B of M as a normal bundle valued symmetric 2-form

$$B : T_x(M) \times T_x(M) \to T_x(M)^{\perp} \text{ at each } x \in M.$$

Then for any $X, Y, Z, W \in X(M)$, we have

$$(\tilde{K}(X, Y) \circ B)(Z, W) = K^{\perp}(X, Y)(B(Z, W)) - B(K(X, Y) Z, W) - B(Z, K(X, Y) W)$$

Putting $Y = Z = \xi$, and using Lemma 2.1 and (1.8), we get

$$(K(X,\ \xi).\ B)(\xi, W) = K^{\perp}(X, \xi)(B(\xi, W)) - B(K(X, \xi)(\xi, W) - B(\xi, K(X, \xi) W) = -\ B(X, W).$$

Then it follows that

Proposition 2.1—Let M be an invariant submanifold of a conformal K-contact Riemannian manifold $\bar{M}$. Then M is minimal if any only if

$$\sum_{i=1}^{2m+1} (\tilde{K}(V_i, \xi).\ B)(\xi, V_i) = 0, \text{ where } (V_1, \ldots, V_{2m+1}) \text{ is a frame in } T_x(M).$$

Proposition 2.2—Let M be an invariant submanifold of a conformal K-contact Riemannian manifold $\bar{M}$. Then M is totally geodesic if and only if $\tilde{K}(X, \xi).\ B = 0$ for any $X \in X(M)$.

From Propositions 2.1 and 2.2, we have

Proposition 2.3—Let M be an invariant submanifold of a conformal K-contact Riemannian manifold $\bar{M}$. Then M is totally geodesic if and only if its second fundamental form B is covariantly constant.

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ASSOCIATED WEBER INTEGRAL TRANSFORMS OF ARBITRARY ORDERS

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In this paper we establish an inversion formula for the Weber-Orr transforms $W_{\mu,\nu}$ [;] of order (μ, ν), and derive many known results as special cases.

1. Introduction

In recent years there have been several applications to elasticity of a formula originally due to Weber, which gives an expression of a continuous function f as a repeated integral (Erdelyi *et al.*[1], p. 74)

$$f(x) = \int_0^\infty \frac{C_\nu(xt, t)}{J_\nu^2(t) + Y_\nu^2(t)} \, t\,dt \int_1^\infty C_\nu(st, t)\, sf(s)\, ds \qquad ..(1.1)$$

where

$$C_\nu(\alpha, \beta) = J_\nu(\alpha) Y_\nu(\beta) - J_\nu(\beta) Y_\nu(\alpha).$$

J_ν and Y_ν being the usual Bessel functions of order ν. This formula was rediscovered by Orr, and Titchwarsh[2] gave a rigorous proof of the formula (1.1) in the following form.

If $x > a > 0$, and $\sqrt{s} f(s)$ is summable in the infinite interval (a, ∞) and $f(s)$ is of bounded variation in a neighbourhood of $t = x$, then for any real ν,

$$\int_0^\infty \frac{C_\nu(xt, at)}{J_\nu^2(at) + Y_\nu^2(at)} \, t\,dt \int_a^\infty C_\nu(st, sa)\, sf(s)\, ds = \tfrac{1}{2}\{f(x+) + f(x-)\}.$$

This is referred to as the Weber-Orr expansion theorem. Alternatively, we restate the above result as follows.

Theorem 1.1—If

(i) $\left| \int_a^\infty \sqrt{x} f(x)\, dx \right| < \infty$

(ii) f continuous on (a, ∞), and

(iii) $$\overset{\Delta}{f}(\xi) = \int_a^\infty R_{\nu,\nu}(\xi;\rho,a)\,\rho f(\rho)\,d\rho \equiv W_{\nu,\nu}[f(\rho);\xi] \qquad ...(1.2)$$

$$f(\rho) = \int_0^\infty \frac{R_{\nu,\nu}(\xi:\rho\ a)}{J^2(\xi a) + Y^2(\xi a)}\,\xi\,\overset{\Delta}{f}(\xi)\,d\xi \equiv W^{-1}_{\nu,\nu}[\hat{f}(\xi);\rho] \qquad ...(1.3)$$

where

$$R_{\nu,\nu}(\xi;\rho,a) = J_\nu(\xi\rho)\,Y_\nu(\xi a) - J_\nu(\xi a)\,Y_\nu(\xi\rho). \qquad ...(1.4)$$

The equations (1.2) and (1.3) define the operators $W_{\nu,\nu}$ [;] and $W^{-1}_{\nu,\nu}$ [;] as the transform and inverse Weber-Orr tronsforms respectively. We shall call (ν, ν) as the order of the operator $W_{\nu,\nu}$ [;]. Recently, associated Weber-Orr transforms of the form $W_{\nu-1,\nu}$ [;] and $W_{\nu-2,\nu}$ [;] were introduced in Krajewsk and Olesiak[3], where it is proved that for $k = 1$ and 2, if

$$\overset{\Delta}{f}(\xi) = W_{\nu-k,\nu}[f(\rho);\xi] = \int_a^\infty R_{\nu-k,\nu}(\xi;\rho,a)\,\rho\, f(\rho)\,d\rho$$

then

$$f(\rho) = W^{-1}_{\nu-k\nu}[\hat{f}(\xi);\rho] = \int_0^\infty \frac{R_{\nu-k,\nu}(\xi;\rho,a)}{J^2_\nu(\xi a) + Y^2_\nu(\xi a)}\,\xi\,\hat{f}(\xi)\,d\xi$$

where

$$R_{\nu-k,\nu}(\xi;\rho,a) = J_{\nu-k}(\xi\rho)\,Y_\nu(\xi a) - J_\nu(\xi a)\,Y_{\nu-k}(\xi\rho).$$

The object of this present note is to introduce the Associated Weber-Orr transform $W_{\mu,\nu}$ [;] of arbitrary order (μ, ν) and establish their inverse transforms $W^{-1}_{\mu,\nu}$ [;] for general values of the parameters μ and ν. Thus generalizing the Weber-Orr transformation, $W_{\nu,\nu}$ [;] and its inverse transformation, $W_{\nu,\nu}$ [;]. All the known results can easily be derived as special cases of our general formulae.

2. Preliminaries

Many properties of the Weber-Orr transforms resemble those of Hankel transforms. For instance, assuming that, for an arbitrary function f, $\left|\int_a^\infty \sqrt{x}\, f(x)\,dx\right| < \infty$, we have from the definition (1.2)

$$\overset{\Delta}{f}(\xi) = W_{\mu,\nu}[f(\rho);\xi] = \int_a^\infty R_{\mu,\nu}(\xi,\rho,a)\,\rho\, f(\rho)\,d\rho \qquad ...(2.1)$$

where

$$R_{\mu,\nu}(\xi, \rho, a) = J_\mu(\xi\rho)\, Y_\nu(\xi a) - J_\nu(\xi a)\, Y_\mu(\xi\rho). \qquad \ldots(2.2)$$

Now integrating by parts the right-hand side of (2.1), we immediately obtain the following result :

$$W_{\mu+1,\nu}\left[\rho^{\mu}\frac{\partial}{\partial\rho}(\rho^{-\mu} f(\rho)); \xi\right] = -\xi W_{\mu,\nu}[f(\rho); \xi] - af(a)\, R_{\mu+1,\nu}(\xi, a, a). \qquad \ldots(2.3)$$

The special cases can easily be derived, using the fact that (Watson[4], p. 46).

$$R_{\nu-1,\nu}(\xi, a, a) = -R_{\nu,\nu-1}(\xi; a, a) = -\frac{2}{\pi\xi a}$$

and

$$R_{\nu,\nu}(\xi; a, a) = 0$$

$$W_{\nu,\nu-1}\left[\rho^{\nu-1}\frac{\partial}{\partial\rho}(\rho^{1-\nu} f(\rho)); \xi\right] = -\xi W_{\nu-1,\nu-1}[f(\rho); \xi] - \frac{2}{\pi\xi} f(a)$$

and

$$W_{\nu,\nu}\left[\rho^{\nu-1}\frac{\partial}{\partial\rho}(\rho^{1-\nu} f(\rho); \xi\right] = -\xi W_{\nu-1,\nu}[f(\rho); \xi].$$

Further, if B_μ denotes the Bessel differential operator

$$B_\mu f(\rho) = \frac{\partial^2 f}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial f}{\partial\rho} - \frac{\mu^2}{\rho^2} f$$

then

$$W_{\mu,\nu}[B_\mu f(\rho); \xi] = -\xi^2 W_{\mu,\nu}[f(\rho); \xi] - a^{1-\mu} R_{\mu,\nu}(\xi; a, a)\frac{\partial}{\partial\rho}(\rho^{\mu} f(\rho))\,|_{\rho=a} + \xi a f(a)\, R_{\mu-1,\nu}(\xi; a, a).$$

The last formula can be established if we write

$$B_\mu f(\rho) = \rho^{\mu-1}\frac{\partial}{\partial\rho}\left(\rho^{1-2\mu}\frac{\partial}{\partial\rho}(\rho^{\mu} f(\rho))\right)$$

and integrate by parts the integral defining the transform $W_{\mu,\nu}[B_\mu f(\rho); \xi]$, given by (2.1). As a special case of (2.4) if we let $\mu = \nu$, then

$$W_{\nu,\nu}[B_\nu f(\rho); \xi] = -\xi^2 W_{\nu,\nu}[f(\rho); \xi] - \frac{2}{\pi} f(a).$$

3. The Transform $W_{\nu-\alpha,\nu}[;]$, $\alpha > 0$

Lemma 3.1—If $0 < \alpha < \frac{1}{2}\nu + \frac{3}{4}$, then

$$R_{\nu-\alpha,\nu}(\xi;\rho,a)=\frac{2^{1-\alpha}}{\Gamma(\alpha)}\rho^{\nu-\alpha}\xi^{\alpha}\int_{\rho}^{\infty} x^{1-\nu}(x^2-\rho^2)^{\alpha-1}R_{\nu,\nu}(\xi;x,a)\,dx \qquad ...(3.1)$$

where

$$R_{\mu,\nu}(\xi,\rho,a)=J_{\mu}(\xi\rho)Y_{\nu}(\xi a)-J_{\nu}(\xi a)Y_{\mu}(\xi\rho).$$

This result follows immediately by making use of the standard integrals involving Besel functions J_ν, Y_ν (Erdelyi *et al.*[5], pp. 25, 104).

Lemma 3.1—If

(i) $0<\alpha<\frac{1}{2}\nu+\frac{3}{4}\ \nu>-\frac{1}{2}$

(ii) $\int_a^\infty |x^{\alpha+1/2}f(x)|\,dx<\infty$

(iii) $$F(x)=\frac{2^{1-\alpha}}{\Gamma(\alpha)}x^{\alpha-\nu}\int_a^x \rho^{1+\nu-\alpha}(x^2-\rho^2)^{\alpha-1}f(\rho)\,d\rho \qquad ...(3.2)$$

then

$$\int_a^\alpha x^{1/2\alpha}F(x)\,dx=\frac{2^{-\alpha}\Gamma(\frac{1}{2}\nu+\alpha-\frac{1}{4})}{\Gamma(\frac{1}{2}\nu+\frac{1}{4})}\int_0^\infty x^{1/2+\alpha}f(x)\,dx. \qquad ...(3.3)$$

PROOF : Now, from (3.2)

$$\int_a^\infty x^{1/2-\alpha}F(x)\,dx=\frac{2^{1-\alpha}}{\Gamma(\alpha)}\int_a^\infty x^{1/2\nu}\,dx\int_a \rho^{1+\nu-\alpha}(x^2-\rho^2)^{\alpha-1}f(\rho)\,d\rho.$$

By changing the order of integration, valid due to absolute convergence, we have

$$\int_a^\infty x^{1/2\alpha}F(x)\,dx=\frac{2^{1-\alpha}}{\Gamma(\alpha)}\int_a^\infty \rho^{1+\nu-\alpha}f(\rho)\,d\rho\int_\rho^\infty x^{1/2\nu}(x^2-\rho^2)^{\alpha-1}\,dx$$

$$=\frac{2^{-\alpha}\Gamma(\frac{1}{2}\nu+\alpha-\frac{1}{4})}{\Gamma(\frac{1}{2}\nu+\frac{1}{4})}\int^\infty \rho^{1/2+\alpha}f(\rho)\,d\rho$$

by evaluating the x-integral, provided $0<\alpha<\frac{1}{2}\nu+\frac{1}{4}$. Now $\alpha\neq\frac{1}{2}\nu+\frac{1}{4}$, because $\nu>-\frac{1}{2}$, therefore by analytic continuation the result holds for the extended range, $0<\alpha<\frac{1}{2}\nu+\frac{3}{4}$, as well.

Corollary 1—

$$\left|\int_a^\infty x^{1/2-\alpha}F(x)\,dx\right|\leqslant\kappa\int_a^\infty|\rho^{1/2+\alpha}f(\rho)|\,d\rho<\infty$$

where κ denotes a constant.

Theorem 3.1—If

(i) $0 < \alpha < \frac{1}{2}\nu + \frac{3}{4}$, $\nu > \frac{1}{2}$

(ii) $\int_a^\infty | x^{1/2+\alpha} f(x) | \, dx < \infty$

(iii) $\hat{f}(\xi) = W_{\nu-\alpha,\nu} [f(\rho); \xi]$,

then

$$x^{-\alpha} F(x) = W^{-1}_{\nu,\nu} [\xi^{-\alpha} \hat{f}(\xi); x]$$

where

$$x^{-\alpha} F(x) = \frac{2^{1-\alpha}}{\Gamma(\alpha)} x^{-\nu} \int_a^x \rho^{1+\nu-\alpha} (x^2 - \rho^2)^{\alpha-1} f(\rho) \, d\rho.$$

PROOF : By the definition of Weber-Orr transform (1.2), we have,

$$\hat{f}(\xi) = \int_a^\infty R_{\nu-\alpha,\nu} (\xi; \rho, a) \rho f(\rho) \, d\rho.$$

The integral exists and in fact absolutely convergent due to (ii). Now using the representation of the function $R_{\nu-\alpha,\nu}$ a given in Lemma 3.1

$$\hat{f}(\xi) = \frac{2^{1-\alpha}}{\Gamma(\alpha)} \xi^{\alpha} \int_a^\infty \rho^{1+\nu-\alpha} f(\rho) \, d\rho \int_\rho^\infty x^{1-\nu} (x^2 - \rho^2)^{\alpha-1} R_{\nu,\nu} (\xi; x, a) \, dx.$$

Changing the order of integration, valid because of absolute convergence, and making use of the representation of the function F, we obtain

$$\xi^{-\alpha} \hat{f}(\xi) = \int_a^\infty R_{\nu,\nu} (\xi; x, a) x^{1-\alpha} F(x) \, dx$$

$$= W_{\nu,\nu} [x^{-\alpha} F(x); \xi].$$

Due to Corollary 1 of Lemma 3.2

$$\left| \int_a^\infty x^{1/2-\alpha} F(x) dx \right| < \infty$$

therefore the inversion formula (1.3) of Theorem 1.1, can be applied to give

$$x^{-\alpha} F(x) = W^{-1}_{\nu,\nu} [\xi^{-\alpha} \hat{f}(\xi); x]$$

as required.

Note that the result is also valid where $\alpha = 0$, if we assume that in this case $F(x) = f(x)$. Our ultimate aim is to find an inversion formula for the Associated Weber-Orr transform of the type $W_{\nu-\alpha,\nu}[;]$, $\alpha > 0$. That is, given the function $\hat{f}$, to find the unknown function f, whereas Theorem 3.1 only enables us to find F defined by

$$x^{-\alpha} F(x) = \frac{2^{1-\alpha}}{\Gamma(\alpha)} x^{-\nu} \int_a^x \rho^{1+\nu-\alpha} (x^2 - \rho^2)^{\alpha-1} f(\rho)\, d\rho. \qquad \ldots(3.4)$$

To retrieve f, we need to invert the integral equation (3.4). This we shall do next, but first we define (Sneddon[6], § 3) an operator $I_{\eta,\alpha}(a, x; 2)$ as

$$I_{\eta,\alpha}(a, x; 2) f(t) = \frac{2}{\Gamma(\alpha)} x^{-2(\eta+2)} \int_a^x x^{2\eta+1} (x^2 - t^2)^{\alpha-1} f(t)\, dt \text{ if } \alpha > 0 \qquad \ldots(3.5)$$

and

$$I_{\eta,\alpha}(a, x; 2) f(t) = x^{-(2\eta+2\alpha+1)} D_x^k \{x^{2k+2\eta+2\alpha+1} I_{\eta,\alpha+k}(a, x; 2) f(t)\} \qquad \ldots(3.6)$$

if

$$\alpha < 0,\ 0 \leqslant \alpha + k < 1,\ D_x = \tfrac{1}{2} \frac{d}{dx} \frac{1}{x} \text{ and } k = 1, 2, 3, \ldots$$

Also let. $I_{\eta,0}(a, x; 2)$ to be the identity operator. This operator is a trivial generalization of the Erdelyi-Kober operator $I_{\eta,\alpha}$ (Sneddon[6], § 2). It can be readily shown that

$$I^{-1}_{\eta,\alpha}(a, x; 2) = I_{\eta+\alpha,-\alpha}(a, x; 2) \qquad \ldots(3.7)$$

defining the inverse operator $I^{-1}_{\eta,\alpha}(a, x; 2)$.

Lemma 3.3—If

$$F(x) = \frac{2^{1-\alpha}}{\Gamma(\alpha)} x^{\alpha-\nu} \int_a^x \rho^{1+\nu-\alpha} (x^2 - \rho^2)^{\alpha-1} f(\rho)\, d\rho$$

then

$$f(x) = \frac{2^{1+\alpha}}{\Gamma(\alpha)} x^{\alpha-\nu-1} D_x^k \left\{ x \int_a^x t^{\nu-\alpha+1} (x^2 - t^2)^{k-\alpha-1} F(t)\, dt \right\}$$

where $\alpha > 0$, $0 \leqslant k - \alpha < 1$, $k = 1, 2, \ldots$

PROOF : Using the definition (3.5) of the operator $I_{\eta,\alpha}$, we write

$$F(x) = \left(\frac{x^2}{2}\right)^\alpha I_{1/2(\nu-\alpha),\alpha}(a, x; 2) f(t), \alpha > 0.$$

Hence due to (3.6), (3.7) we have

$$f(x) = I^{-1}_{1/2(\nu-\alpha),\alpha} \quad (a, x; 2)\left[\left(\frac{2}{t^2}\right)^\alpha F(t)\right]$$

$$= I_{1/2(\nu+\alpha),-\alpha}(a, x; 2)\left[\left(\frac{2}{t^2}\right)^\alpha F(t)\right] \qquad ...(3.8)$$

$$= x^{\alpha-\nu-1} D_x^k \left\{ x^{2k+\nu-\alpha+1} I_{1/2(\nu+\alpha),k-\alpha}(a, x; 2)\left[\left(\frac{2}{t^2}\right)^\alpha F(t)\right]\right\}$$

$$= \frac{2^{1+\alpha}}{\Gamma(k-\alpha)} x^{\alpha-\nu-1} D_x^k \left\{ x \int_a^x t^{\nu-\alpha+1}(x^2 - t^2)^{k-\alpha-1} F(t)\, dt \right\}$$

as desired, where $0 \leqslant k - \alpha < 1$, $k = 1, 2, 3, \ldots$, and $\alpha > 0$.

As a special case, if we set $\alpha = k$, $k = 1, 2, 3, \ldots$, and use the fact that $I_{\eta,0}$ is the identity operator, from (3.8), we obtain,

$$f(x) = x^{k-\nu-1} D_x^k \{2^\alpha x^{\nu-k+1} F(x)\}$$

$$= x^{k-\nu-1}\left(\frac{d}{dx}\frac{1}{x}\right)^k (x^{\nu-k+1} F(x)) = x^{k-\nu}\left(\frac{1}{x}\frac{d}{dx}\right)(x^{\nu-k} F(x)).$$

Hence a useful result.

Lemma 3.4—

$$x^{\nu-k} f(x) = \left(\frac{1}{x}\frac{d}{dx}\right)(x^{\nu-k} F(x)) \qquad ...(3.9)$$

where f and F are defined in Lemma 3.3 above.

Corollary—$F(x) = O(x^{k-3/2-\epsilon})$, $x \to \infty$ iff $f(x) = O(x^{-k3/2\epsilon})$, $x \to \infty$.

The results of Lemmas 3.3 and 3.4 are also valid when $\alpha = 0$, since then $F(x) = f(x)$ throughont. We shall, also, need the following result.

Lemma 3.5—Let $R_{\mu,\nu}(\xi; x, a)$ be defined by (2.2), then

$$\left(\frac{1}{x}\frac{d}{dx}\right)^k \{x^\nu R_{\nu,\nu}(\xi; x, a)\} = \xi^k x^{\nu-k} R_{\nu-k,\nu}(\xi, x, a),\ k = 1, 2, 3, \ldots.$$

This follows directly from the differentiation properties of the Bessel function J_μ and Y_μ, [Watson[4], Chap 3].

Theorem 3.2—If

(i) $0 < k < \frac{1}{2}\nu + \frac{3}{4}$, $k = 1, 2, 3, \ldots$, $\nu > -\frac{1}{2}$.

(ii) $\int_a^\infty | x^{k+1/2} f(x) | \, dx < \infty$

(iii) $\hat{f}(\xi) = W_{\nu-k,\nu}[f(\rho); \xi]$

then

$$f(\rho) = W^{-1}_{\nu-k,\nu} [\hat{f}(\xi); \rho].$$

PROOF : According the assertion of Theorem 3.1, we immediately obtain

$$x^{-k} F(x) = W^{-1}_{\nu,\nu} [\xi^{-k} \hat{f}(\xi); x], \qquad \ldots(3.10)$$

where

$$F(x) = \frac{2^{1-k}}{\Gamma(k)} \, x^{k-\nu} \int_a^x \rho^{1+\nu-k} (x^2 - \rho^2)^{k-1} f(\rho) \, d\rho.$$

Or, using (1.3), the definition of the operator $W^{-1}_{\nu,\nu}$ [;], (3.10) gives,

$$x^{-k} F(x) = \int_0^\infty \frac{R_{\nu,\nu}(\xi; x, a)}{J_\nu^2(\xi a) + Y_\nu^2(\xi a)} \, \xi^{1-k} \hat{f}(\xi) \, d\xi.$$

Now applying the operator $\left(\frac{1}{x} \frac{d}{dx} \right)^k x^\nu$ to both sides of the last equation, we have

$$\left(\frac{1}{x} \frac{d}{dx} \right)^k (x^{\nu-k} F(x)) = \left(\frac{1}{x} \frac{d}{dx} \right)^k (x^\nu) \int_0^\infty \frac{R_{\nu,\nu}(\xi; x, a)}{J_\nu^2(\xi a) + Y_\nu^2(\xi a)} \, \xi^{1-k} \hat{f}(\xi) \, d\xi. \qquad \ldots(3.11)$$

The left-hand side gives $x^{\nu-k} f(x)$ due to (3.9) of Lemma 3.4. Next, we bring the differential operator inside the integral sign and making use of Lemma (3.5), the right hand side of (3.11) becomes

$$x^{\nu-k} \int_0^\infty \frac{R_{\nu-k,\nu}(\xi; x, a)}{J_\nu^2(\xi a) + Y_\nu^2(\xi a)} \, \xi \hat{f}(\xi) \, d\xi. \qquad \ldots(3.12)$$

Note that bringing the differential operators inside the integral is justified because $|R_{\nu-k,\nu}(\xi; x, a)|$ is bounded and the resulting integral (3.12) is uniformly convergent for all $x > a$. Hence equation (3.11), ultimately gives, the required result

$$f(x) = \int_0^\infty \frac{R_{\nu-k,\nu}(\xi; x, a)}{J_\nu^2(\xi a) + Y_\nu^2(\xi a)} \xi \hat{f}(\xi)\, d\xi$$

$$= W^{-1}_{\nu-k,\nu}[\hat{f}(\xi); x]$$

the associa ted Weber-Orr inverse transform. The special case $k = 0$, gives us the W eber-Orr transformation of Theorem 1.1.

4. The Transform $W_{\nu+\alpha,\nu}[;]$, $\alpha > 0$

Lemma 4.1—If $0 < \alpha < \nu + \frac{3}{2}$, then

$$R_{\nu,\nu}(\xi; x, a) = \frac{2^{1-\alpha}}{\Gamma(\alpha)} x^\nu \xi^\alpha \int_x^\infty \rho^{1-\nu-\alpha} (\rho^2 - x^2)^{\alpha-1} R_{\nu+\alpha,\nu}(\xi; \rho, a)\, d\rho \qquad \ldots(4.1)$$

where

$$R_{\mu,\nu}(\xi; \rho, a) = J_\mu(\xi\rho) Y_\nu(\xi a) - J_\nu(\xi a) Y_\mu(\xi\rho).$$

Usin g the standard integrals [Erdelyi *et al.*[5], pp. 25, 104] involving the Bessel functions J_μ and Y_μ, the result can be established quite easily.

Lemma 4.2—If

(i) $\alpha > 0$, $\alpha \neq n + \frac{3}{4} - \frac{1}{2}\nu$, $n = 0, 1, 2, 3, \ldots$

(ii $\int_a^\infty |x^{1/2-\alpha} f(x)|\, dx < \infty$

(iii) $f(x) = \dfrac{2^{1-\alpha}}{\Gamma(\alpha)} x^{-(\nu+\alpha)} \displaystyle\int_a^x \rho^{1+\nu+\alpha} (x^2 - \rho^2)^{\alpha-1} F(\rho)\, d\rho$

then

$$\int_a^\infty x^{1/2-\alpha} f(x)\, dx = \frac{2^{-\alpha}\,\Gamma(\frac{3}{4} - \frac{1}{2}\nu - \alpha)}{\Gamma(\frac{3}{4} - \frac{1}{2}\nu)} \int_a^\infty \rho^{1/2+\nu} F(\rho)\, d\rho.$$

The method of proof is similar to the one used for Lemma 3.2.

Corollary—

$$\left| \int_a^\infty \rho^{1/2+\nu} F(\rho)\, d\rho \right| < K \int_a^\infty | x^{1/2-\alpha} f(x) |\, dx < \infty.$$

where K denotes a constant.

Theorem 4.1—If

(i) $0 < \alpha < \nu + \frac{3}{2}$, $\alpha \neq n + \frac{3}{4} - \frac{1}{2}\nu$

(ii) $\int_a^\infty | x^{1/2\alpha} f(x) |\, dx < \infty$

(iii) $\hat{f}(\xi) = W_{\nu+\alpha,\nu}[f(\rho); \xi]$

then

$$x^\alpha F(x) = W^{-1}_{\nu,\nu}[\xi^\alpha \hat{f}(\xi); x] \qquad \text{...(4.2)}$$

where

$$f(\rho) = \frac{2^{1-\alpha}}{\Gamma(\alpha)} \rho^{-(\nu+\alpha)} \int_a^\rho x^{1+\nu+\alpha} (\rho^2 - x^2)^{\alpha-1} F(x)\, dx. \qquad \text{...(4.3)}$$

PROOF : Now

$$\hat{f}(\xi) = W_{\nu\ \alpha,\nu}[f(\rho); \xi]$$

$$= \int_a^\infty R_{\nu+\alpha,\nu}(\xi; \rho, a)\, \rho\, f(\rho)\, d\rho.$$

By substituting the value of $f(\rho)$ from (4.3) and changing the order of integration, justified because of absolute convergence, we have

$$\hat{f}(\xi) = \frac{2^{1-\alpha}}{\Gamma(\alpha)} \int_a^\infty x^{1+\nu+\alpha} F(x)\, dx \int_x^\infty \rho^{1-\nu-\alpha} (\rho^2 - x^2)^{\alpha-1}$$

$$R_{\nu+\alpha,\nu}(\xi. \rho, a)\, d\rho = \xi^{-\alpha} \int_a^\infty R_{\nu,\nu}(\xi; x, a)\, x^{1+\alpha} F(x)\, dx$$

due to Lemma 4.1

Hence,

$$\xi^\alpha \hat{f}(\xi) = \int_a^\infty R_{\nu,\nu}(\xi; x, a)\, x^{1+\alpha} F(x)\, dx$$

$$= W_{\nu,\nu}[x^\alpha F(x); \xi].$$

And by applying the result Theorem 1.1, since $| \int_a^\infty x^{1/2+\alpha} F(x)\, dx | < \infty$, due to the corollary of Lemma 4.2, we have,

$$x^\alpha F(x) = W^{-1}_{\nu,\nu} [\xi^\alpha \hat{f}(\xi); x],$$

as required.

Alternatively, making use of the operator $I_{\eta,\alpha}$, we obtain, from (4.2) and (4.3)

$$f(\rho) = \left(\frac{\rho}{2}\right)^\alpha I_{1/2\nu,\alpha}(a, \rho; 2)\, [x^\alpha F(x)]$$

$$= \left(\frac{\rho}{2}\right)^\alpha I_{1/2\nu,\alpha}(a, \rho; 2)\, [W^{-1}_{\nu,\nu} [\xi^\alpha \hat{f}(\xi); x]]$$

$$= \left(\frac{\rho}{2}\right)^\alpha I_{1/2\nu,\alpha}(a, \rho; 2) \left[\int_0^\infty \frac{R_{\nu,\nu}(\xi; x, a)}{J^2_\nu(\xi a) + Y^2_\nu(\xi a)}\, \xi^{1+\alpha} \hat{f}(\xi)\, d\xi \right]$$

$$= \left(\frac{\rho}{2}\right)^\alpha \int_0^\infty \frac{I_{1/2\nu,\alpha}(a, \rho; 2([R_{\nu,\nu}(\xi; x, a)}{J^2_\nu(\xi a) + Y^2_\nu(\xi a)}\, \xi^{1+\alpha} \hat{f}(\xi)\, d\xi.$$

Or,

$$f(\rho) = \int_0^\infty \frac{\tilde{R}(\xi, \rho, a)}{J^2_\nu(\xi a) + Y^2_\nu(\xi a)}\, \xi^{1+\alpha} \hat{f}(\xi)\, d\xi \qquad \ldots(4.4)$$

where

$$\tilde{R}(\xi, \rho, a) = \left(\frac{\rho}{2}\right)_\alpha I_{1/2\nu,\alpha}(a, \rho; 2\, [R_{\nu,\nu}(\xi; x, a)]. \qquad \ldots(4.5)$$

Hence, we restate Theorem 4.1 as follows.

Theorem 4.1 (a)—If the assumptions (i) to (ii) of Theorem 4.1 are satisfied, then the unknown function f is given by eqn. (4.4).

If we let $\alpha = 0$ then $f(x) = F(x)$ and $\tilde{R} = \tilde{R}_{\nu,\nu}$, and we have the Weber-Orr transformation of Theorem 1.1.

Next we shall consider a special case when $\alpha = k$, $k = 1, 2, 3, \ldots$. Then from (4 5),

$$\tilde{R}(\xi, \rho, a) = \left(\frac{\rho}{2}\right)^k I_{1,2\nu,k}\,(a, \rho; 2)\, [R_{\nu,\nu}(\xi; x, a)]$$

(*equation continued on p.* 1137)

$$= \frac{2^{1-k}}{\Gamma(k)} \rho^{-(\nu+k)} \int_a^\rho x^{1+\nu} (\rho^2 - x^2)^{k-1} R_{\nu,\nu}(\xi; x, a)\, dx$$

$$= - \rho^{-(\nu+k)} \sum_{m=1}^{k} \frac{2^{m-k} a^{\nu+m}}{(k-m)!\, \xi^m} (\rho^2 - a^2)^{k-m} R_{\nu+m,\nu}(\xi; a, a)$$

$$+ \xi^{-k} R_{\nu+k,\nu}(\xi; \rho, a),$$

by integrating by parts k-times.

Now substituting the above expression $\widetilde{R}$ in (4.4), we have

$$f(\rho) = - \rho^{-(\nu+k)} \sum_{m=1}^{k} \frac{2^{m-k}}{(k-m)!} a^{\nu+m} (\rho^2 - a^2)^{k-m}$$

$$\int_0^\infty \frac{R_{\nu+m,\nu}(\xi; a, a)}{J_\nu^2(\xi a) + Y_\nu^2(\xi a)} \xi^{1+k-m} \hat{f}(\xi)\, d\xi$$

$$+ \int_0^\infty \frac{R_{\nu+k,\nu}(\xi; \rho, a)}{J_\nu^2(\xi a) + Y_\nu^2(\xi a)} \xi \hat{f}(\xi)\, d\xi.$$

Hence the following general result,

Corollary—

$$f(\rho) = - \rho^{-(\nu+k)} \sum_{m=1}^{k} \frac{2^{m-k}}{(k-m)!} a^{\nu+m} (\rho^2 - a^2)^{k-m} W^{-1}_{\nu+m,\nu}$$

$$[\xi^{k-m} \hat{f}(\xi); a] + W^{-1}_{\nu+k,\nu} [\hat{f}(\xi); \rho], \qquad \ldots(4.6)$$

whenever

$$\hat{f}(\xi) = W_{\nu+k,\nu}[f(\rho); \xi],\ k = 1, 2, 3, \ldots.$$

Ih particular if, $k = 1$, the unknown function f is given by

$$f(\rho) = -\left(\frac{a}{\rho}\right)^{\nu+1} W^{-1}_{\nu+1,\nu} [\hat{f}(\xi); a] + W^{-1}_{\nu+1,\nu} \hat{f}(\xi); \rho].$$

where

$$\hat{f}(\xi) = W_{\nu+1,\nu}[f(\rho); \xi].$$

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ON THE GENERATING FUNCTIONS AND PARTIAL SUMS OF THE FOURIER SERIES

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The author has obtained some asymptotic expressions for partial sums of a Fourier series. The results show that certain conditions as imposed on the generating functions of the Fourier series in an earlier paper for a Riesz summability and for the convergence of the series are indeed not only sufficient but necessary as well.

1. INTRODUCTION

Let L denote the space of all 2π-periodic functions which are Lebesgue-integrable over $[0, 2\pi]$ and let $s_n(f, x)$ be the nth partial sum of the Fourier series of $f \in L$ at a point x. For real numbers x, s and d we write

$$\phi(t) = \tfrac{1}{2}\{f(x+t) + f(x-t) - 2s\} \qquad \ldots(1.1)$$

$$\phi_1(t) = (1/t)\int_0^t \phi(u)\, du \qquad \ldots(1.2)$$

$$P(t) = \phi(t) - \phi_1(t) \qquad \ldots(1.3)$$

and for a specified function F,

$$\mathscr{P}_d(F) = \{f \in L \mid \phi_1 \in L \text{ and } \lim_{t \to 0+} t^{-1} F(1/t) \int_0^t |P(u)|\, du = d\}. \qquad \ldots(1.4)$$

In an attempt to improve the well-known Hardy-Littlewood criterion (see Hardy and Littlewood[6]) for the convergence of Fourier series at a point x, we[4] proved the following :

Theorem A—Let $\phi_1(t) = o(1)\ (t \to 0+)$. $\qquad \ldots(1.5)$

Then in order that $(s_n(f; x)) \in (R, \exp(\omega^\alpha), \beta)$ to s, where $0 < \alpha < 1$, $\beta > 0$ and $f \in \mathscr{P}_d(\log)$, it is sufficient that

$$f \in \mathscr{P}_0(\log) \qquad \ldots(1.6)$$

In the same paper, we[4] have also proved the following theorem which improves an earlier result due to Mohapatra[8] :

Theorem B—Let (1.5) hold. Then in order that $(s_n(f; x)) \in (R, \exp\{(\log \omega)^{\Delta}\}, \beta)$ to s, where $f \in \mathscr{P}_d$ (log log), $\beta > 0$ and $\Delta > 0$ however large, it is sufficient that

$$f \in \mathscr{P}_0 \text{ (log log).} \qquad \ldots(1.7)$$

Wang[9] proved that the condition (1.5) alone was not sufficient to ensure summability of $(s_n(f; x))$ by either method. In fact he[9] gave an example of an even function satisfying

$$\phi_1(t) = o\left\{\left(\log \frac{1}{t}\right)^{-1}\right\} \quad (t \to 0+) \qquad \ldots(1.8)$$

whose Fourier series at $x = 0$ diverges. On the other hand the condition (1.8) is not necessary which follows from Izuml *et al.*[7] (see also Bary[1], p. 285). Now the question arises as to whether or not the conditions (1.6) and (1.7) are necessary whenever $f \in \mathscr{P}_d(F)$ satisfies (1.5). In this note we answer this question in affirmative. In fact we prove a general result from which we deduce that if, for $f \in \mathscr{P}_d$ (log),

$$\lim_{t \to 0+} P(t) \log(1/t) = d \neq 0$$

then $s_n(f; x) \sim -d \log\log n$ $(n \to \infty)$, that is the Fourier series of f at x diverges to ∞ for $d < 0$ and hence it cannot be summable by any regular summability method. Consequently the condition

$$P(t) = o(1/\log(1/t)) \quad (t \to 0+)$$

and hence (1.6) is necessary whenever $f \in \mathscr{P}_d$ (log) satisfies (1.5). Similarly, it follows that (1.7) is necessary for $f \in \mathscr{P}_d$ (log log) satisfying (1.5).

We now state the results which we intend to prove.

Theorem 1—Suppose F is a positive and increasing function defined on $(0, \infty)$ and is such that

$$\alpha_n = \int_{1/\pi}^{n} (t\,F(t))^{-1}\,dt \to \infty, \text{ as } n \to \infty. \qquad \ldots(1.9)$$

Then in order that, for $f \in \mathscr{P}_d(F)$

$$s_n(f; x) = o(\alpha_n) \quad (n \to \infty) \qquad \ldots(1.10)$$

it is necessary and sufficient that

$$f \in \mathscr{P}_0(F). \qquad \ldots(1.11)$$

To prove the necessity part of Theorem 1, we first prove.

Theorem 2—Let F and α_n be the same as defined in Theorem 1 and let

$$\lim_{t \to 0+} P(t)\,F(1/t) = d \neq 0. \qquad \ldots(1.12)$$

Then

$$s_n(f; x) \sim -d\alpha_n \qquad \ldots(1.13)$$

We shall write

$$K_n(t) = \frac{\sin(n+\frac{1}{2})t}{2\sin\frac{1}{2}t} - \sum_{m=1}^{n} \frac{\sin mt}{mt}$$

and shall also make use of the following :

$$\sum_{m=1}^{n} \frac{\sin mt}{m} = \tfrac{1}{2}(\pi - t) + O\{(nt)^{-1}\} \; (0 < t < 2\pi) \qquad \ldots(1.14)$$

$$\sum_{m=1}^{\infty} \frac{\sin mt}{m} = \tfrac{1}{2}(\pi - t) \; (0 < t < 2\pi). \qquad \ldots(1.15)$$

The result (1.15) is well-known (e.g. see Bromwich[2], p. 356) and (1.14) may be obtained from it.

2. Proof of Theorem 1

The necessity part of the theorem follows from Theorem 2 so we prove only the sufficiency part.

Proceeding as in Chandra[5], we get

$$s_n(f; x) - s - \frac{1}{\pi}\int_0^{\pi} \phi_1(t)\,dt = \frac{2}{\pi}\int_0^{\pi} P(t)\,K_n(t)\,dt$$

$$= \frac{2}{\pi} J, \text{ say.}$$

Let $\epsilon > 0$ be given. Then there exists $\delta = \delta(\epsilon) > 0$ such that

$$\int_0^t |P(u)|\,du < \epsilon\, t/F(1/t) \text{ for } 0 < t \leqslant \delta < 1. \qquad \ldots(2.1)$$

Then for $n > \delta^{-1}$, we write

$$J = \int_0^{1/n} + \int_{1/n}^{\delta} + \int_{\delta}^{\pi} = J_1 + J_2 + J_3, \text{ say.}$$

We now observe that for all $t \in (0, \pi)$

$$|K_n(t)| \leqslant 3n + 1$$

therefore, by (2.1),

$$|J_1| \leqslant (3n+1) \int_0^{1/n} |P(u)|\, du < 4\,\epsilon/F(n). \qquad \ldots(2.2)$$

Also for all n and uniformly in $t \in (0, \pi)$

$$K_n(t) = O(t^{-1}). \qquad \ldots(2.3)$$

Thus by using (2.3) and the fact that $\delta > 0$ is fixed, we get

$$J_3 = O(1) \int_\delta^\pi t^{-1} |P(t)|\, dt = O(1). \qquad \ldots(2.4)$$

Once again, by using (2.3), we get

$$J_2 = O(1) \int_{1/n}^{\delta} t^{-1} \; |P(t)|\, dt. \qquad \ldots(2.5)$$

Integrating by parts and using (2.1), we get

$$\int_{1/n}^{\delta} t^{-1} |P(t)|\, dt < 2\epsilon/F(1/\delta) + \epsilon \int_{1/n}^{\delta} (tF(1/t))^{-1}\, dt$$

$$= \epsilon\,[2/F(1/\delta) + \alpha_n]$$

by (1.9).

Thus collecting the results and letting $n \to \infty$, we get

$$s_n(f; x) = s + \frac{1}{\pi}\int_0^\pi \phi_1(t)\, dt + o(\alpha_n) = o(\alpha_n)$$

since $\phi_1(t) \in L$ and hence the integral $(1/\pi)\int_0^\pi \phi_1(t)\, dt$ is finite.

This completes the proof of Theorem 1.

3. Proof of Theorem 2

As in Theorem 1,

$$s_n(f; x) = s + \frac{1}{\pi}\int_0^\pi \phi_1(t)\, dt + \frac{2}{\pi} J \qquad \ldots(3.1)$$

where

$$\frac{2}{\pi} J = \frac{2}{\pi}\int_0^\pi \left[P(t)\, F\left(\frac{1}{t}\right) - d\right] \frac{K_n(t)}{F(1/t)}\, dt + \frac{2d}{\pi}\int_0^\pi \frac{K_n(t)}{F(1/t)}\, dt. \qquad \ldots(3.2)$$

Now, since $P(t)\,F(1/t) - d = o(1)$ as $t \to 0+$, therefore proceeding as in Theorem 1, we may obtain that

$$\frac{2}{\pi}\int_0^{\pi} \{P(t)\,F(1/t) - d\}\,\frac{K_n(t)}{F(1/t)}\,dt = o(\alpha_n)\ (n \to \infty). \qquad \ldots(3.3)$$

We have

$$\frac{2d}{\pi}\int_0^{\pi} \frac{K_n(t)}{F(1/t)}\,dt = \frac{2d}{\pi}\int_0^{\pi} \frac{1}{F(1/t)}\,\frac{\sin(n+\frac{1}{2})\,t}{2\sin\frac{1}{2}t}\,dt$$

$$-\frac{2d}{\pi}\int_0^{\pi} \frac{1}{F(1/t)}\left(\sum_{m=1}^{n} \frac{\sin mt}{mt}\right)dt.$$

The first integral on the right and the integral

$$-\frac{2d}{\pi}\int_0^{1/n} \frac{1}{F(1/t)}\left(\sum_{m=1}^{n} \frac{\sin mt}{mt}\right)dt$$

are bounded for all $n \geqslant 1$ and also by (1.14)

$$-\frac{2d}{\pi}\int_{1/n}^{\pi} \frac{1}{F(1/t)}\left(\sum_{m=1}^{n} \frac{\sin mt}{mt}\right)dt$$

$$= -\frac{2d}{\pi}\int_{1/n}^{\pi} (tF(1/t))^{-1}\,\{\tfrac{1}{2}(\pi - t)\}\,dt$$

$$+ O(n^{-1})\int_{1/n}^{\pi} (t^2\,F(1/t))^{-1}\,dt$$

$$= -d\,\alpha_n + (2d/\pi)\int_{1/n}^{\pi} (F(1/t))^{-1}\,dt + O(1)$$

$$= -d\,\alpha_n + O(1).$$

Therefore, collecting the above results, we get

$$\frac{2d}{\pi}\int_0^{\pi} \frac{K_n(t)}{F(1/t)}\,dt = -d\,\alpha_n + O(1). \qquad \ldots(3.4)$$

Combining (3.1) through (3.4), we get

$$s_n(f;x) = s + (1/\pi)\int_0^{\pi}\phi_1(t)\,dt - d\,\alpha_n + O(1)$$

$$= -\,d\,\alpha_n + o(\alpha_n)$$

as $n \to \infty$.

This completes the proof of Theorem 2.

Remark : We remark that if for some $f \in L$

$$\lim_{t \to 0+} P(t)\,F(1/t) \qquad \ldots(3.5)$$

oscillates between some finite real numbers then Theorem 2 is not applicable. In fact it may happen that Fourier series of such an $f \in L$ may converge. For example, let f be an even function and $x = 0$, $s = 0$. Then $\phi(t) = f(t)$. Suppose, for $k > \pi e^2$,

$$\phi(t) = \frac{\sin\left(\log\log \frac{k}{t}\right)}{\log\log \frac{k}{t}} - \frac{\cos\left(\log\log \frac{k}{t}\right)}{\log \frac{k}{t}\,\log\log \frac{k}{t}} \quad (0 < t \leqslant \pi)$$

$$= \alpha(t) - \beta(t), \text{ say.}$$

Then it is easy to verify that

$$P(t) = \frac{1}{t}\int_0^t \frac{\sin\left(\log\log \frac{k}{u}\right)}{\log \frac{k}{u}\left(\log\log \frac{k}{u}\right)^2}\,du - \beta(t).$$

Hence for $F(1/t) = \log(k/t)\log\log(k/t)$

$$P(t)\,F(1/t) = -\cos\left(\log\log \frac{k}{t}\right)$$

$$+\, t^{-1}F(1/t)\int_0^t \frac{\sin\left(\log\log \frac{k}{u}\right)}{\log \frac{k}{u}\left(\log\log \frac{k}{u}\right)^2}\,du.$$

Thus limit in (3.5) oscillates between 1 and -1 but, as it follows from Chandra[3], the Fourier series of this f converges to zero.

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UNSTEADY MOTION OF A SEMI-INFINITE CONDUCTING LIQUID BY A SUDDENLY APPLIED VELOCITY ON ITS SURFACE

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The motion set up in a semi-infinite incompressible viscous conducting liquid by a suddenly applied velocity over a circular area on the free surface of the liquid in presence of magnetic fields acting radially and axially has been obtained. Exact solutions are obtained for both the cases and are presented in the forms of infinite series and infinite integrals.

1. INTRODUCTION

Edward[1] investigated the duct flow of a conducting fluid under circular and radial magnetic fields. Globe[2] solved the problem for a complete annulus under a radial field. Such a field can be produced in a liquid by a line source. Goble[2] and Elco *et al.*[3] pointed out how such a field can be generated in practice. Sengupta and Mahapatra[4] considered the problem of a semi-infinite medium of conducting liquid set in motion by an impulsive velocity on its surface.

In the present paper it is proposed to consider the unsteady rotational motion set up in a semi-infinite medium of viscous incompressible conducting liquid by a suddenly applied velocity within a circular area of the free surface in presence of a radial and axial magnetic fields. In fact, the liquid is contained in the annular region between two circular co-axial cylinders, the inner cylinder having an infinitesimal small circular section and the outer cylinder having a very large circular cross-section. The axis of Z is taken along the common axis of the cylinders and it points into the medium. Solutions are obtained in the forms of infinite series and infinite integrals. Numerical results for the velocity are shown in graphical forms.

2. BASIC EQUATIONS AND BOUNDARY CONDITIONS

Problem I : The Medium is under a Radial magnetic Field

Let (r, θ, z) be the cylindrical coordinates of a point in the liquid having the origin on its surface. Let us suppose that the medium is under the action of a radial magnetic field H_0/r. If (u, v, w) be the velocity components in (r, θ, z) directions, then for rotationally symmetric motion of the liquid $u = w = 0$ and $v = v(r, z, t)$. The linearized equation of motion, in this case, is

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{\partial^2 v}{\partial z^2} - \frac{n^2}{r^2} v = \frac{1}{c^2}\frac{\partial v}{\partial t} \qquad \text{...(2.1)}$$

where

$$c^2 = \frac{\mu}{\rho},\ n^2 = 1 + \frac{\sigma \mu_e^2 H_0^2}{\mu}.$$

μ is the coefficient of viscosity, ρ the density, σ the conductivity, μ_e the mag netic permeability and H_0 is constant.

Problem 2 : The Medium is under an Axial Magnetic Field

Here we suppose that the liquid is acted on by an axial magnetic field H_0 instead of a radial one. In this case the linearised equation of motion is

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{\partial^2 v}{\partial z^2} - \left(q^2 + \frac{1}{r^2}\right) v = \frac{1}{c^2}\frac{\partial v}{\partial t} \qquad \text{...(2.2)}$$

where

$$q^2 = \frac{\sigma \mu_e^2 H_0^2}{\mu}.$$

Boundary conditions—We suppose that the rotational motion of the conducting liquid is set up by prescribing suddenly applied velocity within a circu lar area of the free surface. Thus the boundary conditions for both the problems are

$$v = 0 \text{ as } z \to \infty$$

$$v = f(r)\, H(t) \text{ when } z = 0 \qquad \text{...(2.3)}$$

where

$$f(r) = \epsilon r \text{ when } r \leqslant a$$

$$= 0 \text{ when } r > a \qquad \text{...(2.4)}$$

and $H(t)$ is the Heavislde unit function.

3. Solutions of the Problem

Problem 1—Let us suppose that a solution of the differential equation (2.1) can be taken in the form

$$\phi(\xi, z, t) = \int_0^\infty v(r, z, t)\, J_n(r\xi)\, dr.$$

Then the use of this transform to eqn. (2.1) gives

$$\left(\frac{d^2}{d z^2} - \xi^2\right)\phi = \frac{1}{c^2}\frac{\partial \phi}{\partial t} \qquad \text{...(3.1)}$$

provided that $r\,v \to 0$ as $r \to 0$ and ∞.

Also if we assume that a bar over a function denote its Laplace transform with p as parameter then the equation (3.1) leads to

$$\left(\frac{d^2}{dz^2} - \xi^2 - \frac{p}{c^2}\right)\bar{\phi}\,(\xi, z, p) = 0 \qquad \ldots(3.2)$$

provided $\phi = 0$ at $t = 0$.

Solution of the equation (3.2) finite for $r \to \infty$ is

$$\bar{\phi}\,(\xi, z, p) = A\,(\xi, p) \exp\left(-\frac{z}{c}\sqrt{c^2\,\xi^2 + p}\right). \qquad \ldots(3.3)$$

Now, we express the function $f\,(r)$ given in (2.4) as a Fourier-Bessel integral in the form

$$f\,(r) = \int_0^\infty \xi\, J_n\,(r\xi)\,[\int_0^\infty y\, f\,(y)\, J_n\,(y\theta)\, dy]\, d\xi$$

and replacing $f\,(y)$ by its actual form given in (2.4) and then converting the result to a series we obtain

$$f\,(r) = 2\epsilon a^2 \int_0^\infty J_n\,(r\xi) \sum_{m=0}^{\infty} \frac{(-1)^m\,(\xi a/2)^{n+2m+1}}{m!\,(n+2m+3)\,\Gamma\,(n+2m+1)}\, dr. \ldots(3.4)$$

Hence, with the aid of (3.3) and (3.4) and the boundary conditon (2.3) we get

$$A\,(\xi, p) = \frac{2\epsilon\, a^2}{p} \sum_{m=0}^{\infty} \frac{(-1)^m\,(\xi a/2)^{n+2m+1}}{m!\,(n+2m+3)\,\Gamma\,(n+2m+1)}. \qquad \ldots(3.5)$$

Substituting this value of A in (3.3) and using Laplace inversion theorem we get the expression for $\phi\,(\xi, z, t)$ which leads to

$$v\,(r, z, t) = \epsilon\, a^2 \sum_{m=0}^{\infty} \frac{(-1)^m\, a^{n+2m+1}}{2^{n+2m+1}\, m!\,(n+2m+3)\,\Gamma\,(n+2m+1)}$$

$$\times \int_0^\infty \xi^{n+2m+1}\left[e^{-\xi z} \operatorname{erfc}\left(\frac{z}{2c\sqrt{t}} - c\xi\sqrt{t}\right)\right.$$

$$\left. + e^{\xi z} \operatorname{erfc}\left(\frac{z}{2c\sqrt{t}} + c\xi\sqrt{t}\right)\right] J_n\,(\xi\, r)\, d\xi. \qquad \ldots(3.6)$$

Problem 2—Here we assume a solution of the differential equation (2.2) in the form

$$v = \int_0^\infty \psi\,(\xi, z, t)\, J_1\,(\xi r)\, d\xi \qquad \ldots(3.7)$$

where ψ satisfies the differential equation

$$\frac{\partial^2 \psi}{\partial z^2} = (q^2 + \xi^2)\,\psi + \frac{1}{c^2}\frac{\partial \psi}{\partial t}. \qquad \text{...(3.8)}$$

Applying Laplace transform to eqn. (3.8) subject to $\psi = 0$ at $t = 0$ and the boundary condition (2.3) we get after necessary calculations

$$\bar{\psi}(\xi, z, p) = \frac{\epsilon a^2}{p} J_2(\xi a) \exp\left(-\sqrt{c^2 \eta^2 + p}\right) \qquad \text{...(3.9)}$$

where $\eta^2 = q^2 + \xi^2$. Laplace inversion of (3.9), with the help of (3.7) gives

$$v(r, z, t) = \tfrac{1}{2}\,\epsilon\, a^2 \int_0^\infty \Big[e^{-\eta z} \operatorname{erfc}\left(\frac{z}{2c\sqrt{t}} - c\eta\sqrt{t}\right)$$

$$+ e^{\eta z} \operatorname{erfc}\left(\frac{z}{2c\sqrt{t}} + c\,\eta\sqrt{t}\right)\Big] J_2(\xi a)\, J_1(\xi r)\, d\xi. \qquad \text{...(3.10)}$$

4. Numerical Results

It is clear from (3.6) and (3.10) that in either case, the initial response of the velocity is infinitely small and it increases with time giving a steady response after an infinite interval of time. Taking mercury as the relevant conducting fluid, the nature of the velocity has been shown in Fig. 1 (for problem 1) and in Fig. 2 (for problem 2). For numerical calculations we have chosen $a = 1$, $z = 1$, $r = 0.5$, $c = 0.33215$ and $n = 1, 2, 3$ $n = 1$ gives the nature for non-MHD case while $n = 2,3$ show the distribution for MHD case. It is seen that the magnetic field decreases the velocity for both the problems.

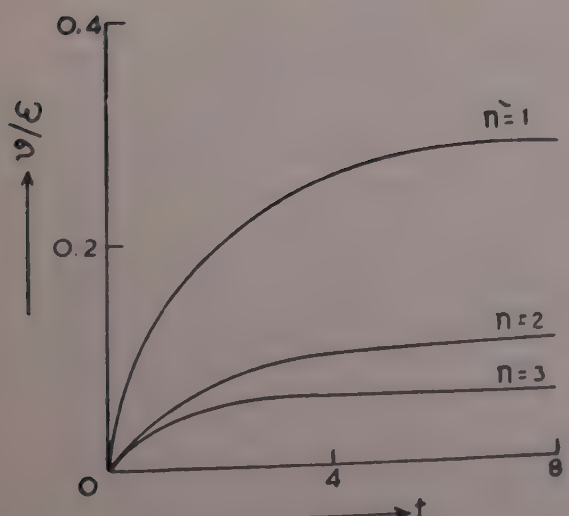

Fig. 1.

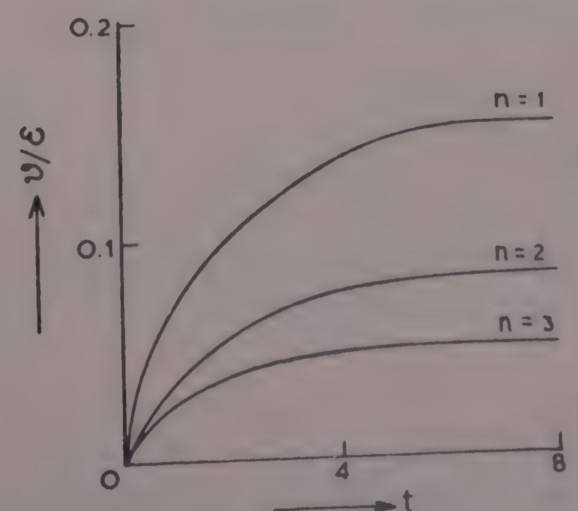

Fig. 2.

It may be noted in this connection that the integrals involved in (3.6) and (3.10) seem to be difficult to evaluate analytically and therefore they are obtained numerically by Filon's method[6].

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ON HYDROMAGNETIC TURBULENT SHEAR FLOW

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Hydromagnetic turbulent shear flow of viscous incompressible, electrically conducting fluid between two infinite porous horizontal planes in presence of a uniform transverse magnetic field has been studied by the semi-empirical method. The expressions for the mean distributions for velocity and magnetic field have been obtained when the surfaces of the channel are non-conducting and conducting. Numerical results are shown graphically.

1. Introduction

It is known that a turbulent state eventually results from the instability of laminar flow in a celestial body like the sun, earth etc., and there are many ways of producing turbulence, for example by thermal instability or by a flow of air through a wind tunnel[1]. Turbulent shear flow of an incompressible viscous fluid between two parallel planes and through a circular pipe has been studied by Pai[2,3] by the semi-empirical approach suggested by Kampe de Ferriet[4]. These theoretical results agree with the experimental results of Laufer[5] and Nikurdse[6]. Jain's[7] solutions for hydromagnetic turbulent shear flow between two parallel non-permeable planes are also in close conformity with the experimental results of Murgatroyd[8]. Mehta and Balasubramanyam[9] also solved a similar type of problem.

In the present paper, it is proposed to study the hydromagnetic turbulent shear flow of an incompressible viscous electrically conducting fluid between two horizontal parallel permeable planes in presence of a uniform transverse magnetic field by the semi-empirical method of Kampe de Ferriet[4]. Two cases are considered : (i) the surfaces of the channel are non-conducting and (ii) the surfaces are conducting of the same conductivity. Due to non-availability of the relevant experimental data, assumptions are made regarding the numerical values of the constants. The expressions for the velocity and the magnetic field are obtained and their natures are shown in graphical forms for both the cases.

2. Governing Equations

We consider the fully developed steady state hydromagnetic turbulent shear flow of an incompressible viscous electrically conducting fluid between two uniformly

porous parallel planes at a distance $2L$ apart. Let the x-axis be in the direction of the flow parallel to the planes, the y-axis normal to the planes and the z-axis transverse to both x and y. Let the middle plane be $y = 0$ and the hydromagnetic flow variables are functions of y only. The planes of the channel are now $y = \pm L$.

Neglecting displacement currents, the hydromagnetic equations in e.m. units are[9]

$$v_j \frac{\partial v_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 v_i}{\partial x_j \partial x_j} + h_j \frac{\partial h_i}{\partial x_j} - \frac{1}{2}\frac{\partial h_j^2}{\partial x_i} \quad ...(2.1)$$

$$v_j \frac{\partial h_i}{\partial x_j} - h_j \frac{\partial v_i}{\partial x_j} = \nu_H \frac{\partial^2 h_i}{\partial x_j \partial x_j} \quad ...(2.2)$$

$$\frac{\partial v_i}{\partial x_i} = 0 \quad ...(2.3)$$

$$\frac{\partial h_i}{\partial x_i} = 0 \quad ...(2.4)$$

where $(i, j = 1, 2, 3)$, $(x_1, x_2, x_3) = (x, y, z)$, $h_i = H_i/\sqrt{4\pi\rho}$, H_i are the magnetic field intensity components, v_i velocity components, p pressure, ρ density, ν kinetic viscosity, ν_H $(= 1/4\,\pi\sigma)$ magnetic diffusivity and σ is the electrical conductivity.

Let the flow be composed of a mean motion with superimposed random fluctuations and eqns. (2.1) to (2.4) are satisfied by the instantaneous flow variables, which may be expressed as

$$f = \bar{f} + f' \quad ...(2.5)$$

where $\bar{f}$ and f' denote the mean and flutuating parts of the flow variable respectively.

Substituting (2.5) into equs. (2.1) to (2.4) we get[9]

$$\bar{v}_j \frac{\partial \bar{v}_i}{\partial x_j} - \bar{h}_j \frac{\partial \bar{h}_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x_i} + \nu \frac{\partial^2 \bar{v}_i}{\partial x_j \partial x_j} - \frac{\partial b_{ij}}{\partial x_j} - \frac{1}{2}\left(\frac{\partial \bar{h}_j^2}{\partial x_i} + \frac{\partial \overline{h_j'^2}}{\partial x_i}\right) \quad ...(2.6)$$

$$\bar{v}_j \frac{\partial \bar{h}_i}{\partial x_j} - \bar{h}_j \frac{\partial \bar{v}_i}{\partial x_j} = \nu_H \frac{\partial^2 \bar{h}_i}{\partial x_j \partial x_j} + \frac{\partial a_{ij}}{\partial x_j} \quad ...(2.7)$$

$$\frac{\partial \bar{v}_i}{\partial x_i} = 0 \quad ...(2.8)$$

$$\frac{\partial \bar{h}_i}{\partial x_i} = 0 \quad ...(2.9)$$

where

$$\left.\begin{aligned} a_{ij} &= < v'_i h'_j > - < v'_j h'_i > \\ b_{ij} &= < v'_i v'_j > - < h'_i h'_j > \end{aligned}\right\} \qquad \ldots(2.10)$$

and < > denote average value.

Let us take $\bar{v}_i = \{\bar{v}_x(y), \bar{v}_y(y), 0\}$, $\bar{h}_i = \{\bar{h}_x(y), \bar{h}_y(y), 0\}$, the components of a_{ij} and b_{ij} are functions of y only and assume that there is a uniform transverse magnetic field h_0 perpendicular to the main flow direction. Then equations (2.8) and (2.9) give

$$\bar{v}_y = \text{constant} = v_0, \quad \bar{h}_y = \text{constant} = h_0. \qquad \ldots(2.11)$$

Introducing the non-dimensional quantities

$$\xi = \frac{x}{L}, \eta = \frac{y}{L}, V = \frac{\bar{v}_x}{u^*}, H = \frac{\bar{h}_x}{u^*}$$

$$A_{ij} = \frac{a_{ij}}{u^{*2}}, B_{ij} = \frac{b_{ij}}{u^{*2}}, R^* = \frac{Lu^*}{\nu}, R^*_M = \frac{Lu^*}{\nu_H} \qquad \ldots(2.12)$$

$$R = \frac{Lv_0}{\nu}, \text{ cross-flow Reynold number}$$

$$R_M = \frac{Lh_0}{\nu_H} \text{ Magnetic Reynold number}$$

$$\epsilon = \frac{\nu}{\nu_H}, \tilde{\omega} = \frac{\bar{p} - \bar{p}_0}{\rho u^{*2}}, \chi = \frac{1}{u^{*2}} < \left(h'^2_x + h'^2_y + h'^2_z \right) >$$

where $u^* = \sqrt{\tau/\rho}$ is the reference velocity, τ is the shearing stress on the plane $y = L$ (i.e. on $\eta = 1$) and $\bar{p}_0$ is the reference pressure which is taken as the mean pressure at $\xi = 0$, $\eta = 1$, the hydromagnetic equations (2.6) and (2.7) reduce to

$$\frac{d^2V}{d\eta^2} - R\frac{dV}{d\eta} + \frac{R_M}{\epsilon}\frac{dH}{d\eta} = R^*\left(\frac{\partial\tilde{\omega}}{\partial\xi} + \frac{dB_{xy}}{d\eta}\right) \qquad \ldots(2.13)$$

$$\frac{\partial\tilde{\omega}}{\partial\eta} + \frac{d}{d\eta}\left[B_{yy} + \frac{1}{2}(H^2 + \chi)\right] = 0 \qquad \ldots(2.14)$$

$$\frac{dB_{yz}}{d\eta} = 0, \quad \frac{dB_{zx}}{d\eta} = 0 \qquad \ldots(2.15)$$

$$\frac{d^2H}{d\eta^2} - \epsilon R\frac{dH}{d\eta} + R_M\frac{dV}{d\eta} + R^*_M\frac{dA_{xy}}{d\eta} = 0. \qquad \ldots(2.16)$$

3. Solutions for the Problem

Case I : The Surfaces are Non-conducting

Here the boundary conditions for the problem are

$$V = H = 0 \quad \text{at} \quad \eta = \pm 1 \qquad \text{...(3.1)}$$

$$A_{ij} = B_{ij} = 0 \quad \text{at} \quad \eta = \pm 1 \qquad \text{...(3.2)}$$

$$\tilde{\omega}(\xi, \eta) = 0 \quad \text{at} \quad \xi = 0, \eta = 1. \qquad \text{...(3.3)}$$

Equations (2.14) and (2.15) when integrated subject to the boundary conditions (3.2) and (3.3) lead to

$$A_{yz} = B_{yz} = 0$$

$$\tilde{\omega}(\xi, \eta) + B_{yy} + \tfrac{1}{2}(H^2 + \chi) = A_0 \, \xi \qquad \text{.. (3.4)}$$

where $A_0 = \dfrac{\partial \tilde{\omega}}{\partial \xi}$ is the axial pressure gradient assumed to be given for the flow. From these results we made the same conclusions as in Mehta and Balasubramanyam[9].

In the absence of turbulence $A_{ij} = B_{ij} = 0$ and the solutions of the equations (2.13) and (2.16) satisfying the boundary conditions (3.1) and (3.13) are given by

$$V_l = V_c \left[\epsilon R a_0 + (\epsilon R - \alpha) a_1 e^{\alpha \eta} + (\epsilon R - \beta) a_2 e^{\beta \eta} + a_3 - \epsilon R \eta\right] \qquad \text{...(3.5)}$$

and

$$H_l = V_c R_M \left[a_0 + a_1 e^{\alpha \eta} + a_2 e^{\beta \eta}\right] \qquad \text{...(3.6)}$$

where

$$\alpha, \beta = \tfrac{1}{2}\left[R(1 + \epsilon) \pm \sqrt{\{R^2(1 - \epsilon)^2 + 4M^2\}}\right]$$

$$a_0 = \frac{\beta \coth \alpha - \alpha \coth \beta}{\alpha - \beta}$$

$$a_1 = -\frac{\beta \operatorname{cosech} \alpha}{\alpha - \beta}$$

$$a_2 = \frac{\alpha \operatorname{cosech} \beta}{\alpha - \beta}$$

$$a_3 = \frac{\alpha\beta(\coth \beta - \coth \alpha)}{\alpha - \beta} \qquad \text{...(3.7)}$$

$V_c = A_0 R^* / \alpha\beta$ is the characteristic velocity and $M = a h_0 / \sqrt{\nu \nu_H}$ is the Hartmann number.

In the presence of turbulence, $A_{ij}, B_{ij} \neq 0$ and we may get the solution for the mean velocity distribution V_l for the turbulent shear flow compatiable with the corres-

ponding laminar flow with the same characteristic velocity V_c, by assuming V_t of the form

$$V_t = V_c\,[\epsilon R a_0 + (\epsilon R - \alpha)\, a_1\, e^{\alpha\eta} + (\epsilon R - \beta)\, a_2\, e^{\beta\eta} + a_3 - \epsilon R\eta + A_1\,(\eta + 1)^2 + A_2\,(\eta + 1)^m],\quad m > 2 \qquad \text{...(3.8)}$$

satisfying the boundary condition $V_t = 0$ at $\eta = -1$.

Introducing the empirical parameter

$$s = \frac{\tau_t}{\tau_l} = \frac{\left(\dfrac{dV_t}{d\eta}\right)_{\eta=1}}{\left(\dfrac{dV_l}{d\eta}\right)_{\eta=1}} \qquad \text{.. (3.9)}$$

and using the boundary condition $V_t = 0$ at $\eta = 1$ we get

$$A_1 = -\frac{s-1}{2\,(m-2)\,(\alpha-\beta)}\Big[(\alpha-\beta)\,(\alpha\beta - \epsilon R) + \epsilon R a_3 + \alpha\beta\,(\alpha \coth\alpha - \beta \coth\beta)\Big]$$

$$A_2 = \frac{s-1}{2^{m-1}\,(m-2)\,(\alpha-\beta)}\Big[(\alpha-\beta)\,(\alpha\beta - \epsilon R) + \epsilon R a_3 + \alpha\beta\,(\alpha \coth\alpha - \beta \coth\beta)\Big]. \qquad \text{...(310)}$$

For turbulent flow $s > 1$ and for laminar flow $s = 1$. The parameters s and m are to be determined experimentally.

Similarly, we assume the magnetic field for the turbulent flow in the form

$$H_t = V_c\, R_M\,[a_0 + a_1\, e^{\alpha\eta} + a_2\, e^{\beta\eta} - \eta + A_3\,(\eta + 1)^2 + A_4\,(\eta + 1)^m],\quad m > 2 \qquad \text{...(3.11)}$$

which satisfies the boundary condition at $\eta = -1$. Introducing the emperical parameter

$$l = \frac{\left(\dfrac{dH_t}{d\eta}\right)_{\eta=1}}{\left(\dfrac{dH_l}{d\eta}\right)_{\eta=1}} \qquad \text{...(3.12)}$$

and using the boundary condition at $\eta = 1$, we get

$$A_3 = -\frac{(l-1)\,(a_3 - \alpha + \beta)}{2\,(m-2)\,(\alpha-\beta)}$$

$$A_4 = \frac{(l-1)\,(a_3 - \alpha + \beta)}{2^{m-1}\,(m-2)\,(\alpha-\beta)}\,. \qquad \text{...(3.13)}$$

For turbulent flow $l > 1$ and for laminar flow $l = 1$. The parameters l and m are to be determined experimentally.

Case II : The Surfaces are Conducting

We assume that the surfaces $\eta = \pm 1$ of the channel are at rest and they are conducting. Then the boundary conditions for the problem are

$$
\begin{aligned}
&V = 0 && \text{at} \quad \eta = \pm 1 \\
&\phi \frac{dH}{d\eta} \pm H = 0 && \text{at} \quad \eta = \pm 1 \\
&A_{ij} = B_{ij} = 0 && \text{at} \quad \eta = \pm 1 \\
&\tilde{\omega} (\xi, \eta) = 0 && \text{at} \quad \xi = 0, \ \eta = 1
\end{aligned}
\qquad \text{...(3.14)}
$$

where ϕ is the conductance ratio.

Proceeding exactly along the same lines as in case I, we find that the solutions for the velocity and the magnetic field for laminar flow are

$$V_l = V_c \left[\epsilon R b_0 + (\epsilon R - \alpha)\, b_1 e^{\alpha\eta} + (\epsilon R - \beta)\, b_2 e^{\beta\eta} + b_3 + 1 - \epsilon R\eta\right] \qquad \text{...(3.15)}$$

$$H_l = V_c R_M \left[b_0 + b_1 e^{\alpha\eta} + b_2 e^{\beta\eta} - \eta\right] \qquad \text{...(3.16)}$$

and for turbulent flow

$$
\begin{aligned}
V_t = V_c \big[&\epsilon R b_0 + (\epsilon R - \alpha)\, b_1 e^{\alpha\eta} + (\epsilon R - \beta)\, b_2 e^{\beta\eta} \\
&+ b_3 + 1 - \epsilon R\eta + B_1 (\eta + 1)^2 + B_2 (\eta + 1)^n
\end{aligned}
\qquad \text{...(3.17}
$$

$$H_t = V_c R_M \left[b_0 + b_1 e^{\alpha\eta} b_2 e^{\beta\eta} - \eta + B_3 (\eta + 1)^2 + B_4 (\eta + 1)^n\right. \qquad \text{...(3.18)}$$

where $n > 2$ and

$$
\begin{aligned}
B_1 &= -\frac{t - 1}{2(\eta - 2)} \left[(\epsilon R - \alpha)\, b_1 \alpha e^{\alpha} + (\epsilon R - \beta)\, b_2 \beta e^{\beta} - \epsilon R \right] \\
B_2 &= \frac{t - 1}{2^{\eta - 2} (\eta - 2)} \left[(\epsilon R - \alpha)\, b_1 \alpha e^{\alpha} + (\epsilon R - \beta)\, b_2 \beta e^{\beta} - \epsilon R \right] \\
B_3 &= -\frac{(q - 1)(\phi n + 2)}{2(\eta - 2)} \left[\alpha b_1 e^{\alpha} + \beta b_2 e^{\beta} - 1 \right] \\
B_4 &= \frac{(q - 1)(\phi n + 1)}{2^{\eta - 1} (\phi + 1)} \left[\alpha b_1 e^{\alpha} + \beta b_2 e^{\beta} - 1 \right]
\end{aligned}
\qquad \text{...(3.19)}
$$

$$b_0 = (\phi + 1) - (\phi\alpha + 1)\, b_1 e^{\alpha} - (\phi\beta + 1)\, b_2 e^{\beta}$$

$$b_1 = \frac{1}{\Lambda} \left[(\phi + 1)(\epsilon R - \beta) \sinh \beta - \epsilon R (\phi\beta \cosh \beta + \sinh \beta) \right]$$

$$b_2 = -\frac{1}{\Lambda}\left[(\phi + 1)(\epsilon R - \alpha)\sinh\alpha - \epsilon R(\phi\alpha\cosh\alpha + \sinh\alpha)\right]$$

$$b_3 = -\epsilon R b_0 - (\epsilon R - \alpha)\, b_1\, e^{\alpha} - (\epsilon R - \beta)\, b_2\, e^{\beta} - 1 + \epsilon R$$

$$\Lambda = (\epsilon R - \beta)\sinh\beta\,(\phi\alpha\cosh\alpha + \sinh\alpha)$$

$$- (\epsilon R - \alpha)\sinh\alpha\,(\phi\beta\cosh\beta + \sinh\beta)$$

and

$$t = \frac{\tau_t}{\tau_l} = \frac{\left(\dfrac{dV_t}{d\eta}\right)_{\eta=1}}{\left(\dfrac{dV_l}{d\eta}\right)_{\eta=1}}, \quad q = \frac{\left(\dfrac{dH_t}{d\eta}\right)_{\eta=1}}{\left(\dfrac{dH_t}{d\eta}\right)_{\eta=1}} = \frac{(H_t)_{\eta=1}}{(H_l)_{\eta=1}} \qquad \ldots(3.20)$$

are the empirical parameters. For turbulent flow $t, q > 1$ and for laminar flow $t, q = 1$.

3. Numerical Results

As the experimental results are not available, we assume for numerical discussions

$$R = 2 \;,\; M = 1 \;,\; R_M = 1.5 \;,\; \epsilon = 0.5 \;,\; A_0 = 1,$$

$$R^* = 3 \;,\; \phi = 0.5 \;,\; m = n = 3 \;,\; s = t = 3 \;,\; l = q = 4.$$

The behaviour of the velocity distribution has been shown in Figure 1 while that of the magnetic field has been exhibited in Figure 2. Continuous curves represent the nature of the entities for turbulent flow while the results for laminar flows are given by the dashed curves.

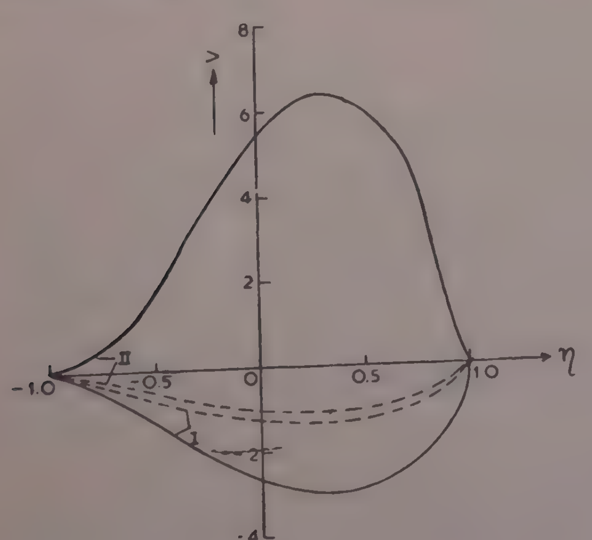

Fig. 1.

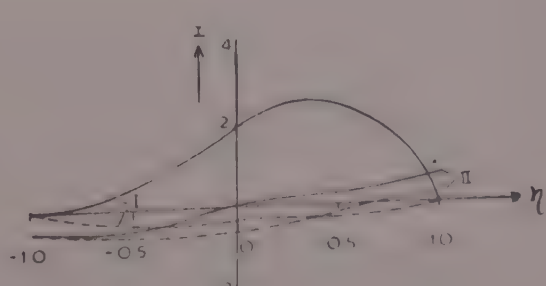

Fig. 2.

Figure 1 shows that the turbulence reduces the velocity for non-conducting walls, but it increases the velocity when the walls are conducting. On the other hand,

Figure 2 sh ows that the turbulence always increases the magnetic field whether the walls are conducting or non-conducting.

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ON THE STABILITY OF COMPRESSIBLE SWIRLING FLOWS

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For a swirling flow $(0, V(r), W(r))$ of an inviscid, compressible fluid we obtain an instability region for subsonic axisymmetric disturbances which depend on the minimum Richardson number J_0, wave number k and the distance between the cylinders. This region reduces to the line $c_i = 0$ as $J_0 \to 1/4 -$ for monotonic axial velocity profile in accord with the Richardson number criterion. An instability region depending on J_0 is also obtained for a class of supersonic axisymmetric disturbances.

1. INTRODUCTION

Many of the results on the linear stability of variable density, inviscid shear flows in the presence of gravity, due to Miles[7], Howard[2], and Chimonas[1] have been shown to possess parallels in stability analyses of axisymmetric disturbances in an inviscid swirling flow between two infinite concentric cylinders in the works of Howard and Gupta[4], Howard[3], and Lalas[6]. Howard and Gupta[4] examined stability of a swirling, incompressible, constant density, inviscid flow $(0, V(r), W(r))$ for which the pressure, axial velocity and azimuthal velocity are functions of radius r only and showed that a sufficient condition for stability is that the minimum Richardson number J_0 is greater than or equal to one quarter. Moreover, they showed that the complex wave velocity c for an unstable mode must lie inside a semicircle in the upper half-plane whose diameter coincides with the range of the axial velocity. Howard[3] investigated the stability of compressible swirling flows to axisymmetric disturbances and showed that a sufficient condition for stability is that $J_0 \geqq \frac{1}{4}$ and an instability region is given by the semicircle in the upper half-plane with the range of the axial velocity as its diameter. However, this instability region given by Howard's semicircle does not depend on the stratification or compressible parameters. In particular, even when $J_0 \geqq \frac{1}{4}$ one still gets a semicircle as the instability region where as the flow is known to be stable in that case.

In this paper we obtain an instability region for subsonic axisymmetric disturbances which depend not only on stratification but also on the wave number and the distance between the cylinders. This region lies inside Howard's semicircle and reduces to the line $c_i = 0$ when $W'_{\min} \neq 0$ and $J_0 \to \frac{1}{4} -$ in accord with the Richardson number criterion. Further we reduce the instability region given by the semicircle

theorem for a class of supersonic axisymmetric disturbances. Similar results have been obtained by Jain and Kochar[5] for incompressible parallel shear flows and by Subbiah and Jain[8] for compressible parallel shear flows.

The stability of compressible swirling flows to all (not necessarily axisymmetric) disturbances has been studied by Lalas[6] and he has proved the Richardson number criterion for stability.

2. Basic Equations and Boundary Conditions

Consider the isentropic flows of an inviscid compressible fluid confined in the annular region $R_1 \leqq r \leqq R_2$. Then the basic equations are

$$\rho\left[\frac{\partial \vec{U}}{\partial t} + \vec{U}.\nabla\vec{U}\right] = -\nabla p \qquad \text{...(1)}$$

$$\frac{\partial \rho}{\partial t} + \vec{U}.\nabla\rho + \rho\nabla.\vec{U} = 0 \qquad \text{...(2)}$$

$$\frac{Ds}{Dt} = 0, \; p = p\,(\rho\; s). \qquad \text{...(3,4)}$$

Here ρ is the density, p the pressure, s the specific entropy, $\vec{U}$ the velocity. The operator $\frac{D}{Dt}$ is defined as usual by

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{U}.\nabla. \qquad \text{...(5)}$$

Equation (4) is the equation of state for the fluid, and in conjunction with (3) gives

$$\frac{\partial p}{\partial t} + \vec{U}.\;\nabla p + \rho a_2^* \;(\nabla.\vec{U}) = 0 \qquad \text{...(6)}$$

where

$$a_*^2 = \left(\frac{\partial p}{\partial \rho}\right)_s. \qquad \text{...(7)}$$

The system of equations is now comprised of (1), (2) and (6) along with the definition of (7).

If we fix the (r, θ, z) coordinate system with r-axis along the radial direction and z-axis along the axial direction, then the flow variables given by

$$\vec{U} = (0, V(r), W(r)), \; \rho = \rho_0(r), \; p = p_0(r) \qquad \text{...(8)}$$

satisfy the governing equations (1) (2) and (6) provided

$$\frac{dp_0}{dr} = \frac{\rho_0 V^2}{r} \qquad \text{...(9)}$$

where $\rho_0(r)$, $V(r)$ and $W(r)$ are arbitrary, twice continuously differentiable functions of r.

We are interested in examining the stability of the basic state (8), (9) to infinitesimal disturbances of the form

$$(\text{function of } r).\ \exp(ikz + im\theta - ikct) \qquad \ldots(10)$$

where the wave numbers k, m are real and $c = c_r + ic_i$ is the complex wave velocity.

Linearization of the basic equations (1), (2) and (6) about a basic flow given by (8), (9) yields the following set of linearised equations for the perturbation quantities p, ρ, u, v, and w:

$$i\rho_0 \Omega u - \frac{2V v \rho_0}{r} - \frac{\rho V^2}{r} = -p' \qquad \ldots(11)$$

$$i\rho_0 \Omega v + \rho_0 \left[V' + \frac{V}{r}\right] u = -\frac{imp}{r} \qquad \ldots(12)$$

$$i\rho_0 \Omega w + \rho_0 u W' = -ikp \qquad \ldots(13)$$

$$i\rho \Omega + u\rho_0' + \rho_0\left[\frac{du}{dr} + \frac{u}{r} + \frac{imv}{r} + ikw\right] = 0 \qquad \ldots(14)$$

$$ip\Omega + up_0' - a_*^2\left[i\rho\Omega + uP_0'\right] = 0 \qquad \ldots(15)$$

where $\Omega = \dfrac{mV}{r} + kW - kc$ and a prime denotes differentiation with respect to r. Eliminating p, ρ, v and w in favour of u from (11) – (15) we obtain

$$\frac{d}{dr}\left\{\frac{\rho_0 \Omega S}{\Delta}\left[\frac{du}{dr} + \frac{u}{r} - u\left(\frac{m}{r\Omega}\left(V' + \frac{V}{r}\right) + \frac{kW'}{\Omega} - \frac{V^2}{ra_*^2}\right)\right]\right\}$$

$$+\left[\frac{2Vm}{r^2\Omega} - \frac{V^2}{ra_*^2}\right]\left\{\frac{\rho_0 \Omega S}{\Delta}\left[\frac{du}{dr} + \frac{u}{r} - u\left(\frac{m}{r\Omega}\left(V' + \frac{V}{r}\right) + \frac{kW'}{\Omega} - \frac{V^2}{ra_*^2}\right)\right]\right\}$$

$$-\rho_0 \Omega u\left[1 - \frac{N^2}{\Omega^2} - \frac{\Phi}{\Omega^2}\right] = 0 \qquad \ldots(16)$$

where

$$S = \frac{r^2}{m^2 + r^2k^2}, \quad \Delta = 1 - \frac{S\Omega^2}{a_*^2} \qquad \ldots(17), (18)$$

$$N^2 = \frac{V^2}{r}\left[\frac{\rho_0'}{\rho_0} - \frac{V^2}{ra_*^2}\right] \qquad \text{...(19)}$$

$$\Phi = \frac{2V(V' + V/r)}{r}. \qquad \text{...(20)}$$

N is the effective Brunt-Väisälä frequency which is assumed to be positive for static stability: Φ is the Rayleigh discriminant.
Define

$$Q = \exp\left[-\int^r \frac{V^2}{ra_*^2}\, dr\right] \qquad \text{...(21)}$$

and

$$F = \frac{u}{\Omega Q}. \qquad \text{...(22)}$$

With these definitions, eqn. (16) becomes

$$\frac{d}{dr}\left[\frac{\rho_0\,\Omega S}{\Delta}\left\{\Omega\,\frac{(rF)'}{r} - \frac{2mV}{r^2}\,F\right\}Q^2\right] + \rho_0\,\frac{2VmS}{r^2\,\Delta}$$
$$\left\{\Omega\,\frac{(rF)'}{r} - \frac{2mV}{r^2}\,F\right\}Q^2 - \rho_0\,Q^2\left[\Omega^2 - \Phi - N^2\right]F = 0 \qquad \text{...(23)}$$

with the associated boundary conditions

$$F(R_1) = 0 = F(R_2). \qquad \text{...(24)}$$

Suppose now that the flow is unstable, so that c has a positive imaginary part and define a new variable

$$G = \Omega^{1/2}\,F. \qquad \text{...(25)}$$

Then equation (23) becomes

$$\left[\frac{\rho_0\,Q^2\,\Omega S}{\Delta}\left(G' + \frac{G}{r}\right)\right]' - \frac{\Omega'^2\,\rho_0\,Q^2\,S}{4\Omega\Delta} + \frac{\Omega'}{2}\,\frac{\rho_0\,Q^2\,S}{\Delta}$$
$$\left[1 - \frac{4mV}{r\,\Omega}\right]\frac{G}{r} - \left\{\frac{S\rho_0\,Q^2}{2\Delta}\left[\Omega' + \frac{4mV}{r^2}\right]\right\}'G$$
$$+ \frac{\rho_0\,Q^2\,S}{\Delta}\,\frac{2m\,V}{r^2}\left[1 - \frac{2mV}{r\Omega}\right]\frac{G}{r} - \rho_0\,Q^2\,\Omega$$
$$\left[1 - \frac{\Phi}{\Omega^2} - \frac{N^2}{\Omega^2}\right]G = 0 \qquad \text{...(26)}$$

and the associated boundary conditions are

$$G(R_1) = 0 = G(R_2). \qquad \text{...(27)}$$

3. Instability Regions

For an unstable mode, the imaginary part of the equation, obtained by multiplying (26) by rG^* (G^* is the complex conjugate of G) and integrating it using (27), gives

$$\int \frac{\rho_0 Q^2}{|\Delta|^2}\Big[\frac{S}{r}\Big(1+\frac{|\Omega|^2 S}{a_*^2}\Big)|(rG)'|^2 - \frac{2\Omega_r S^2}{a_*^2}\Big(\frac{\Omega'}{2}+\frac{2mV}{r^2}\Big)$$

$$\times [G^*(rG)' + G(rG^*)']$$

$$+ \frac{S^2}{a_*^2}\Big(1+\frac{|\Omega|^2 S}{a_*^2}\Big)\Big[\frac{\Omega'}{2}+\frac{2mV}{r^2}\Big]^2 r|G|^2\Big]$$

$$+ \int \rho_0 Q^2 r|G|^2 + \int \frac{\rho_0 Q^2 r}{|\Omega|^2}\Big[(N^2+\Phi) - S\Big(\frac{\Omega'}{2}$$

$$+ \frac{2mV}{r^2}\Big)^2\Big]|G|^2 = 0 \qquad \text{...(28)}$$

where Ω_r is the real part of Ω.

At this stage, we deviate from the analysis of Lalas[6] and rewrite eqn. (28) as

$$\int \frac{\rho_0 Q^2}{|\Delta|^2}\Big[\Big(1+\frac{S\Omega_r^2}{a_*^2}\Big)\Big\{\frac{S}{r}|(rG')|^2 + \frac{S^2}{a_*^2}$$

$$\Big(\frac{\Omega'}{2}+\frac{2mV}{r^2}\Big)^2 r|G|^2\Big\}$$

$$+ \frac{\Omega_i^2}{a_*^2}\Big\{\frac{S}{r}|(rG)'|^2 + \frac{S^2}{a_*^2}\Big(\frac{\Omega'}{2}+\frac{2mV}{r^2}\Big)^2 r|G|^2\Big\}$$

$$- \frac{2\,\Omega_r S^2}{a_*^2}\Big(\frac{\Omega'}{2}+\frac{2mV}{r^2}\Big)(G^*(rG)' + G(rG^*))\Big] + \int \rho_0 r Q^2|G|^2$$

$$+ \int \frac{\rho_0 Q^2 r}{|\Omega|^2}\Big[(N^2+\Phi) - S\Big(\frac{\Omega'}{2}+\frac{2mV}{r^2}\Big)^2\Big]|G|^2 = 0. \qquad \text{...(29)}$$

Since $G(rG^*)' + (G(rG^*)' \leqq 2|G|\,|(rG)'|$, we have

$$\Big(1+\frac{\Omega_r^2 S}{a_*^2}\Big)\Big|\frac{\Omega'}{2}+\frac{2mV}{r^2}\Big|\Big(\frac{S}{a_*^2}\Big)^{1/2} r|G|\,|(rG)'|$$

$$-\Big(\frac{\Omega'}{2}+\frac{2mV}{r^2}\Big)\Omega_r \frac{S}{a_*^2} r[G^*(rG)' + G(rG^*)'] \geqq 0. \qquad \text{...(30)}$$

Therefore

$$\frac{\rho_0 Q^2}{|\Delta|^2}\frac{S}{r}\Big[\Big(1+\frac{S\Omega_r^2}{a_*^2}\Big)\Big\{|(rG)'|^2+\frac{S}{a_*^2}\Big(\frac{\Omega'}{2}+\frac{2mV}{r^2}\Big)^2 r^2|G|^2\Big\}\Big]$$

$$-\frac{2\Omega_r S}{a_*^2} r\Big(\frac{\Omega'}{2}+\frac{2mV}{r^2}\Big)[G^*(rG)'+G(rG^*)']$$

$$=\frac{\rho_0 Q^2 S}{|\Delta|^2 r}\Big[\Big(1+\frac{S\Omega_r^2}{a_*^2}\Big)^{1/2}|(rG)'|-\Big(1+\frac{S\Omega_r^2}{a_*^2}\Big)^{1/2}\Big(\frac{S}{a_*^2}\Big)^{1/2}\Big|\frac{\Omega'}{2}$$

$$+\frac{2mV}{r^2}\Big|r|G|\Big]^2+2\Big(1+\frac{S\Omega_r^2}{a_*^2}\Big)\Big(\frac{S}{a_*^2}\Big)^{1/2}\Big|\frac{\Omega'}{2}+\frac{2mV}{r^2}\Big|$$

$$r|G|\ |(rG)'|\ \frac{2\Omega_r S}{a_*^2} r\Big(\frac{\Omega'}{2}+\frac{2mV}{r^2}\Big)[G^*(rG)'$$

$$+G(rG^*)']\geqq 0 \text{ by (30).}$$

Therefore, for an unstable mode, (29) implies that

$$\int\frac{\rho_0 Q^2}{|\Delta|^2}\frac{S}{r}\Big[\frac{\Omega_i 2}{a_*^2}\Big\{|(rG)'|^2+\frac{S}{a_*^2}\Big(\frac{\Omega'}{2}+\frac{2mV}{r^2}\Big)^2 r^2|G|^2\Big\}\Big]$$

$$+\int\rho_0 Q^2 r|G|^2+\int\frac{\rho_0 Q^2 r}{|\Omega|^2}$$

$$\times\Big[(N^2+\Phi)-S\Big(\frac{\Omega'}{2}+\frac{2mV}{r^2}\Big)^2\Big]|G|^2\leqq 0. \qquad \ldots(31)$$

This is impossible if $(N^2+\Phi)\geqq S\Big[\frac{\Omega'}{2}+\frac{2mV}{r^2}\Big]^2$ that is if

$$(N^2+\Phi)\geqq\Big(\frac{\Omega'}{2}+\frac{2mV}{r^2}\Big)^2\Big|\Big(\frac{m^2}{r^2}+k^2\Big) \qquad \ldots(32)$$

every where in $[R_1, R_2]$.

This proves the following theorem of Lalas[6].

Theorem 1—A sufficient condition for stability is that

$(N^2+\Phi)\geq\Big(\frac{\Omega'}{2}+\frac{2mV}{r^2}\Big)^2\Big|\Big(\frac{m^2}{r^2}+k^2\Big)$ throughout $[R_1, R_2]$.

In an effort to obtain a criterion that does not depend explicitly on the wave numbers, we note that

$$\left[\frac{\Omega'}{2} + \frac{2mV}{r^2} \right]^2 \Big| \left(\frac{m^2}{r^2} + k^2 \right) \leqq \tfrac{1}{4} \left[W'^2 + \left(V' + \frac{3V}{r} \right)^2 \right] \qquad \text{...(33)}$$

and so if

$$N^2 + \Phi - \frac{1}{4} \left[W'^2 + \left(V' - \frac{3V}{r} \right)^2 \right] \geqq 0 \qquad \text{...(34)}$$

the flow would be stable.

Rearranging (34) we obtain

$$N^2 - \frac{1}{4} \left[W'^2 + \left(V' - \frac{V}{r} \right)^2 \right] \geqq 0$$

for stability.

Defining the local Richardson number as $J = N^2 \Big| \left[W'^2 + \left(V' - \frac{V}{r} \right)^2 \right]$, then Theorem 1 means that there is no instability region if $J_0 \geqq \frac{1}{4}$, where $J_0 = \min\limits_{r} J(r)$.

The above result is the Richardson number criterion for compressible swirling flows to all small disturbances obtained by Lalas[6].

If we put $m = 0$ in the above result, then the resultant is the Richardson criterion for swirling flows, for axisymmetric disturbances only, obtained by Howard[3].

The disturbances are classified as subsonic, sonic or supersonic if $| W - c | \begin{matrix} < \\ = \\ > \end{matrix} a_* .$

Theorem 2—The complex wave velocity c for any unstable supersonic axisymmetric disturbance satisfying $c_i^2 > a^2_{*_{max}}$ must be inside the semiellipse region, in the upper half plane given by

$$\left(c_r - \frac{a+b}{2} \right)^2 + c_i^2 \left[1 + \frac{4 J_0}{\left[1 + \left(1 + \frac{5(b-a)^2}{4a^2_{*_{min}}} \right) \sqrt{1 - 4 J_0} \right]^2} \right]$$

$$\leqq \left(\frac{b-a}{2} \right)^2 \qquad \text{...(35)}$$

where $a \leqq W(r) \leqq b$.

PROOF : For axisymmetric disturbances (i. e. for which $m = 0$) the linear stability equation (23) reduces to

$$\frac{d}{dr}\left[\frac{\rho_0 Q^2 a_*^2 (W-c)^2}{(W-c)^2 - a_*^2} \quad \frac{(r F)'}{r}\right]$$

$$+ \rho_0 Q^2 [k^2 (W-c)^2 - \Phi - N^2] F = 0 \qquad \ldots(36)$$

with the boundary conditions

$$F(R_1) = 0 = F(R_2).$$

Mulitiplying (36) by rF^* and integrating it using the boundary conditions and following the standard procedure (Howard[2]), we get

$$\left[\left(c_r - \frac{a+b}{2}\right)^2 + c_i^2 - \left(\frac{b-a}{2}\right)^2\right] \int U(r)\, dr + \int \rho_0 Q^2 r$$

$$(\Phi + N^2) \mid F \mid^2 + \int \frac{\rho_0 Q^2 \mid W-c \mid^4 r^{-1} a_*^2 \mid (rF)' \mid^2}{\mid a_*^2 - (W-c)^2 \mid^2} \leqq 0.. \; (37)$$

where

$$U(r) = \rho_0 Q^2 \left[\frac{r^{-1} a_*^4 \mid (rF)' \mid^2}{\mid a_*^2 - (W-c)^2 \mid^2} + rk^2 \mid F \mid^2\right], \; a \leqq W(r) \leqq b.$$

For supersonic axisymmetric, unstable modes satisfying $c_i^2 / a_{*\,max}^2 > 1$, we have from (31)

$$\int \rho_0 Q^2 \left\{\frac{r^{-1} \mid (rG)' \mid^2}{\left|1 - \frac{(W-c^2)}{a_*^2}\right|^2} + k^2 r \mid G \mid^2\right\} \leqq (1 - 4 J_0)$$

$$\times \int \frac{\rho_0 Q^2 r W'^2 \mid G \mid^2}{4 \mid W-c \mid^2}. \qquad \ldots(38)$$

Let

$$E^2 = \int k \mid W-c \mid U(r)\, dr$$

and

$$B^2 = \int \frac{\rho_0 Q^2 kr \mid W' \mid^2 \mid F \mid^2}{4 \mid W-c \mid \left|1 - \frac{(W-c)^2}{a_*^2}\right|^2}.$$

Now substitute $G = \Omega^{1/2} F$ in (38) and use Schwarz's inequality to get

$$(E - B)^2 \leqq (1 - 4 J_0) \left[1 + \frac{5 (b - a)^2}{4a^2_{*_{\min}}} \right]^2 B^2.$$

This gives

$$E^2 \leqq \left[1 + \left(1 + \frac{5 (b - a)^2}{4a^2_{*_{\min}}} \right) \sqrt{1 - 4 J_0} \right]^2 B^2. \quad \ldots(40)$$

Since, for an unstable supersonic axisymmetric mode with

$$\frac{c_i^2}{a_*^2} > 1, \left| 1 - \frac{(W - c)^2}{a_*^2} \right| > 1$$

we have

$$\int \rho_0 Q^2 r (\Phi + N^2) \mid F \mid^2 \geqq \frac{4 J_0 c_i^2 B^2}{k}$$

$$\geqq \frac{4 J_0 c_i E^2}{k \left[1 + \left(1 + \frac{5 (b - a)^2}{4a^2_{*_{\min}}} \right) \sqrt{1 - 4 J_0} \right]^2}$$

$$\geqq \frac{4 J_0 c_i^2 \int U (r)\, dr}{\left[1 + \left(1 + \frac{5 (b - a)^2}{4a^2_{*_{\min}}} \right) \sqrt{1 - 4 J_0} \right]^2}. \quad \ldots(41)$$

Using this estimate (41) in (37) we get (36). This proves the theorem.

Theorem 3—The complex wave velocity c for any unstable subsonic axisymmetric mode must lie inside the semiellipse type region, in the upper half plane given by

$$\left(c_r - \frac{a + b}{2} \right)^2 + c_i^2$$

$$+ \frac{J_0 W'^2_{\min} (\lambda^2 + k^2) a^2_{*_{\min}} c_i^8 \left[1 - \frac{5 (b - a)^2}{4a^2_{*_{\min}}} \right]^2}{W'^4_{\max} a^6_{*_{\max}} (\frac{1}{4} - J_0) [\frac{1}{2} + 2 \sqrt{\frac{1}{4} - J_0}]^2}$$

$$\leq \left(\frac{b - a}{2} \right)^2 \quad \ldots(42)$$

where

$$\lambda^2 = \frac{\rho_{0\,\min}\, a^2_{*_{\min}}\, \pi^2}{\rho_{0\max}\, a^2_{*_{\max}}\, (y_2 - y_1)^2}.$$

PROOF : Since, for subsonic axisymmetric unstable modes,

$$\frac{(W - c_r)^2 + c_i^2}{a_*^2} \leqq 1$$

we have

$$\frac{c_i^2}{a_*^2} < 1 \text{ in } [R_1, R_2].$$

Therefore, for an unstable subsonic axisymmetric mode, we have from (31)

$$\int \frac{\rho_0 Q^2}{\left| 1 - \frac{(W - c)^2}{a_*^2} \right|^2} \left\{ \frac{c_i^2}{a_*^2} \left(r^{-1} \mid (rG)' \mid^2 + k^2 \left| 1 - \frac{(W - c)^2}{a_*^2} \right|^2 \right.\right.$$

$$\left.\left. \times\, r \mid G \mid^2 \right) \right\} \leqq (\tfrac{1}{4} - J_0) \int \frac{\rho_0 Q^2\, r\, W'^2 \mid G \mid^2}{\mid W - c \mid^2}. \qquad \ldots(43)$$

Let

$$E_1^2 = \int \frac{\rho_0 Q^2\, k\, c_i^2 \mid W - c \mid}{a_*^2 \left| 1 - \frac{(W - c)^2}{a_*^2} \right|^2} \left\{ r^{-1} \mid (rF)' \mid^2 + k^2 \left| 1 - \frac{(W - c)^2}{a_*^2} \right|^2 r \mid F \mid^2 \right\}$$

and

$$B_1^2 = \int \frac{\rho_0 Q^2\, c_i^2\, kr \mid W' \mid^2 \mid F \mid^2}{a_*^2 \left| 1 - \frac{(W - c)^2}{a_*^2} \right|^2 \mid W - c \mid}.$$

Then, proceeding as in the proof of the previous theorem, we get

$$E_1^2 \leqq \frac{a_{*\,\max}^2}{c_i^2}\left[\tfrac{1}{2} + 2\sqrt{\tfrac{1}{4} - J_0}\right]^2 B_1^2 . \qquad \ldots(44)$$

Use of the Rayleigh-Ritz inequality shows that

$$\frac{\displaystyle\int \frac{\rho_0 Q^2 c_i^2 r^{-1} \,|\,(rG)'\,|^2}{a_*^2 \left|1 - \dfrac{(W-c)^2}{a_*^2}\right|^2}}{\displaystyle\int \frac{\rho_0 Q^2 c_i^2 \, r \,|\,G\,|^2}{a_*^2}} \geqq \lambda^2 \qquad \ldots(45)$$

where

$$\lambda^2 = \frac{\rho_{0\min}\, a_{*\,\min}^2 \quad \pi^2}{\rho_{0\max}\, a_{*\,\max}^2 \quad 4\,(y_2 - y_1)^2} . \qquad \ldots(46)$$

Use of (45) in (43) leads to

$$(\lambda^2 + k^2) \leqq (\tfrac{1}{4} - J_0)\, \frac{W'^2_{\max} \quad a_{*\,\max}^2}{c_i^4} . \qquad \ldots(47)$$

If

$$\nu^2 = \frac{\displaystyle\int \frac{\rho_0 \, Q^2 r^{-1} \,|\,(r\,F)'\,|^2}{\left|1 - \dfrac{(W-c)^2}{a_*^2}\right|^2}}{\displaystyle\int \rho_0 \, Q^2 \, r \,|\,F\,|^2}$$

then use of (47) gives

$$\frac{\int U(r)\, dr}{\int \rho_0 \, Q^2 \, r \,|\,F\,|^2} = (\nu^2 + k^2) = \frac{(\nu^2 + k^2)\,(1/4 - J_0)\, a_{*\,\max}^2 \quad W'^2_{\max}}{(\lambda^2 + k^2) c_i^4} \qquad \ldots(48)$$

We have

$$\frac{E_1^2}{B_1^2} = \frac{\displaystyle\int \frac{c_i^2 \mid W - c \mid U(r)}{a_*^2}}{\displaystyle\int \frac{\rho_0 Q^2 r c_i^2 \mid W' \mid^2 \mid F \mid^2}{a_*^2 \left| 1 - \dfrac{(W-c)^2}{a_*^2} \right| \mid W - c \mid}}$$

$$\geqq \frac{c_i^2 \; a_{*\min}^2 \; (\nu^2 + k^2) \left[1 - \dfrac{5(b-a)^2}{4a_{*\min}^2} \right]^2}{W'^2_{\max} \; a^2_{*_{\max}}}$$

which can be rewritten as

$$(\nu^2 + k^2) \leqq \frac{W'^2_{\max} \; a^2_{*_{\max}}}{c_i^2 \; a^2_{*_{\min}} \left[1 - \dfrac{5(b-a)^2}{4a^2_{*_{\min}}} \right]^2} \; \frac{E_1^2}{B_1^2} \; .$$

Using the inequality (44), we get

$$(\nu^2 + k^2) \leqq \frac{\dfrac{W'^2_{\max} \; a^4_{*_{\max}}}{c_i^4 \; a^2_{*_{\min}}} \left[\tfrac{1}{2} + 2\sqrt{\tfrac{1}{4} - J_0}\right]^2}{\left[1 - \dfrac{5(b-a)^2}{4a^2_{*_{\min}}} \right]^2} \; . \qquad \ldots(49)$$

Using this in (48), we have

$$\frac{\int U(r)\, dr}{\int \rho_0 Q^2 r \mid F \mid^2} \leqq \frac{W'^4_{\max} \; a^6_{*_{\max}} \; \left[\tfrac{1}{2} + 2\sqrt{\tfrac{1}{4} - J_0}\right] (\tfrac{1}{4} - J_0)}{c_i^8 \; a^2_{*_{\min}} \; (\lambda^2 + k^2) \left[1 - \dfrac{5(b-a)^2}{4a^2_{*_{\min}}} \right]^2} \; . \qquad \ldots(50)$$

Therefore,

$$\int \rho_0 Q^2 (\Phi + N^2) r \mid F \mid^2 \geqq J_0 W'^2_{\min} \int \rho_0 Q^2 r \mid F \mid^2$$

$$\geqq \frac{J_0 W'^2_{\min} \; (\lambda^2 + k^2) \; a^2_{*_{\min}} \left[1 - \dfrac{5(b-a)^2}{4a^2_{*_{\min}}} \right]^2 c_i^8 \int U(r)\, dr}{W'^4_{\max} \; a^6_{*_{\max}} \; (\tfrac{1}{4} - J_0) \left[\tfrac{1}{2} + 2\sqrt{\tfrac{1}{4} - J_0}\right]^2} \; . \qquad \ldots(51)$$

Use of this in (37), gives (42). This proves the theorem.

In the last two theorems, we have reduced the instability region given by the semicircle theorem. The reduced regions depend on stratification through the minimum Richardson number J_0. The result for subsonic axisymmetric disturbances incorporate not only the stratification but also the wave number and the distance between the cylinders. Furthermore, when $W'_{\min} \neq 0$, $J_0 \to \frac{1}{4} -$ implies $c_i \to 0^+$ in accord with Theorem 1 for axisymmetric disturbances.

4. Concluding Remarks

In the linear stability analysis of inviscid compressible swirling flows between two infinite concentric cylinders we have obtained an instability region for subsonic axisymmetric disturbances which depend on the stratification, wave number and the distance between the cylinders. This region, which lies inside Howard's semicircle, reduces to the line $c_i = 0$ as $J_0 \to \frac{1}{4} -$ for monotonic axial velocity profile in accord with the Richardson number criterion. Furthermore, we have established an instability region depending on the stratification for a class of supersonic disturbances.

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